Formally Smooth Manifolds and Microbundles

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Abstract

This thesis is structured in two parts: in the first part, we introduce a few standard objects such as microbundles, spherical fibrations, spaces of smooth structures and recall important results about them. We also provide a short introduction to smoothing theory and obstruction theory.
In the second part, we use facts from the first part to prove that there is a homotopy equivalence of Postnikov 5-truncations $\mathrm{BSTop}(4)_{\leq 5} \simeq \mathrm{BSO}(4)_{\leq 5} \times K(\mathbb{Z}/2, 4)$. This homotopy equivalence is closely connected to the relation between the spaces of smooth structures $\mathrm{Sm}(M)$ and formally smooth structures $\mathrm{Sm}^f(M)$ on a 4-manifold M. We establish a microbundle analogue of Dold and Whitney's classification of vector bundles over 4-dimensional CW complexes ([DW59]) as an application of the mentioned homotopy equivalence of 5-truncations.

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1 Introduction

Classification of smooth structures on a topological manifold is a classical problem in geometric topology. It is known that in dimensions up to 3, all topological manifolds admit a unique smooth structure up to diffeomorphism. The situation in dimensions 5 and above is more complicated; nevertheless, Kirby and Siebenmann proved a classification theorem which provides a comprehensive description of the space of smooth structures.

Theorem 1.1 ([KS77] 2.3 on page 235). *Let M be a topological manifold without boundary and* $\dim M \neq 4$. *Then there is a homotopy equivalence*

$$\theta : \mathrm{Sm}(M) \to \mathrm{Sm}^f(M).$$

There is a variation of this theorem which also covers manifolds with boundary. We introduce the space of smooth structures $\mathrm{Sm}(M)$ and briefly outline a proof of this theorem in Section 3. The space $\mathrm{Sm}^f(M)$ on the right is substantially simpler and, in some sense, captures only the tangential data of a manifold M. We define spaces of formally smooth and stably formally smooth structures as

$$\mathrm{Sm}^f(M) := \mathrm{Lift}(M \xrightarrow{\mathrm{t}M} \mathrm{BTop}(d) \text{ to } \mathrm{BO}(d)),$$

$$\mathrm{Sm}^{sf}(M) := \mathrm{Lift}(M \xrightarrow{\mathrm{t}M} \mathrm{BTop} \text{ to } \mathrm{BO}),$$

where $\mathrm{t}M : M \to \mathrm{BTop}(d)$ is a classifying map of the tangent microbundle of M which we also introduce in Section 2. More intuitively, the space of formally smooth structures contains information about vector bundle structures on the tangent microbundle of M.

1.1 Motivation and main results

In this thesis our main focus is on the space of smooth structures of manifolds of dimension 4. The theorem of Kirby and Siebenmann does not hold in dimension 4; nevertheless, Freedman and Quinn proved a result which shows that there is still a strong connection between spaces of smooth and formally smooth structures. Let M be a topological 4-manifold, then the pullback of $\mathrm{t}M$ along the collapse map

$$c : M\#nS^2 \times S^2 \to M$$

is stably isomorphic to $\mathrm{t}(M\#nS^2 \times S^2)$ because the stable tangent microbundle of $S^2 \times S^2$ is trivial. Therefore, any stably formally smooth structure on M also induces one on $M\#nS^2 \times S^2$ by pullback.

Theorem 1.2 ([FQ90] Chapter 8.6). *Let M be a topological 4-manifold. For any stably formally smooth structure σ on M there exists a smooth structure Σ on $M\#nS^2 \times S^2$ for some $n \in \mathbb{Z}_{\geq 0}$, such that the pullback of σ along the collapse map*

$$c : M\#nS^2 \times S^2 \to M$$

is isomorphic to the stably formally smooth structure induced by the smooth structure Σ.

Using action maps, one can construct a commutative triangle

$$\begin{array}{ccc} & \mathrm{Homeo}(M) & \\ \swarrow & & \searrow^{\alpha} \\ \mathrm{Sm}(M) & \xrightarrow{\theta} & \mathrm{Sm}^f(M), \end{array} \tag{1.3}$$

which can potentially help to deduce information about $\mathrm{Sm}(M)$ by investigating the map α. It would be a good start if we could understand homotopy groups of $\mathrm{Sm}^f(M)$, this is the main question we attempted to address in this thesis. It is simpler to consider the space of stably formally smooth structures $\mathrm{Sm}^{sf}(M)$, Milgram proved the following theorem.

Theorem 1.4 ([Mil88]). *There is a homotopy equivalence of 7-types* $\mathrm{BSTop}_{\leq 7} \xrightarrow{\theta} \mathrm{BSO}_{\leq 7} \times K(\mathbb{Z}/2, 4)$ *such that the composition*

$$\mathrm{BSO} \to \mathrm{BSTop} \to \mathrm{BSTop}_{\leq 7} \xrightarrow{\theta} \mathrm{BSO}_{\leq 7} \times K(\mathbb{Z}/2, 4) \xrightarrow{\mathrm{pr}_1} \mathrm{BSO}_{\leq 7}$$

is 8-connected, and

$$\mathrm{BSTop} \to \mathrm{BSTop}_{\leq 7} \xrightarrow{\theta} \mathrm{BSO}_{\leq 7} \times K(\mathbb{Z}/2, 4) \xrightarrow{\mathrm{pr}_2} K(\mathbb{Z}/2, 4)$$

corresponds to the Kirby-Siebenmann class.

Thus, for an oriented manifold M with $\mathrm{Sm}^{+,sf}(M) \neq \emptyset$ we have isomorphisms

$$\pi_k \,\mathrm{Sm}^{+,sf}(M) \simeq H^{3-k}(M, \mathbb{Z}/2), \text{ for } k = 0, 1, 2; \tag{1.5}$$

where "+" means that we are taking orientations into account. Combining this with Quinn's result that the stabilization map $\mathrm{Top}(4)/\mathrm{O}(4) \to \mathrm{Top}/\mathrm{O}$ is 5-connected ([FQ90] Theorem 8.7A) we can also compute the zeroth homotopy group of the unstable space

$$\pi_0 \,\mathrm{Sm}^{+,f}(M) \simeq H^3(M, \mathbb{Z}/2).$$

Furthermore, Galvin proved:

Theorem 1.6 ([Gal24]). *Let M be a connected, orientable, closed, smooth 4-manifold, then the map*

$$\pi_0 \,\mathrm{Homeo}(M \# S^2 \times S^2) \xrightarrow{\alpha_*} \pi_0 \,\mathrm{Sm}^{+,f}(M \# S^2 \times S^2) \simeq H^3(M \# S^2 \times S^2, \mathbb{Z}/2)$$

is surjective.

In view of 1.3 we get a lower bound on the size of $\pi_0 \,\mathrm{Sm}(M \# S^2 \times S^2)$. Thus, the proposed approach can potentially also yield interesting elements in the higher homotopy groups of $\mathrm{Sm}(M)$. Unfortunately, we were not able to compute any $\pi_k \,\mathrm{Sm}^f(M)$ for $k > 0$, mainly because we do not know enough about $\mathrm{Top}(4)/\mathrm{O}(4)$. However, we still establish an analogue of Milgram's theorem for $\mathrm{BSTop}(4)$ which is the main technical result of this thesis.

Theorem A (Theorem 6.1)**.** There is a homotopy equivalence of 5-types $\mathrm{BSTop}(4)_{\leq 5} \xrightarrow{\theta} \mathrm{BSO}(4)_{\leq 5} \times K(\mathbb{Z}/2, 4)$ such that the composition

$$\mathrm{BSO}(4) \to \mathrm{BSTop}(4) \to \mathrm{BSTop}(4)_{\leq 5} \xrightarrow{\theta} \mathrm{BSO}(4)_{\leq 5} \times K(\mathbb{Z}/2, 4) \xrightarrow{\mathrm{pr}_1} \mathrm{BSO}(4)_{\leq 5}$$

is 6-connected, and

$$\mathrm{BSTop}(4) \to \mathrm{BSTop}(4)_{\leq 5} \xrightarrow{\theta} \mathrm{BSO}(4)_{\leq 5} \times K(\mathbb{Z}/2, 4) \xrightarrow{\mathrm{pr}_2} K(\mathbb{Z}/2, 4)$$

corresponds to the Kirby-Siebenmann class.

As an application of this theorem, we deduce a microbundle analogue of Dold and Whitney's classification of vector bundles over 4-dimensional CW complexes ([DW59]).

Theorem B (Theorem 7.2)**.** Let X be a topological 4-manifold or a locally finite 4-dimensional simplicial complex with no 2-torsion in $H^4(X, \mathbb{Z})$. Then the maps:

$$\mathrm{Mic}_3^+(X) \xrightarrow{(w_2, \tilde{p}_1)} H^2(X, \mathbb{Z}/2) \times H^4(X, \mathbb{Z}),$$

$$\mathrm{Mic}_4^+(X) \xrightarrow{(w_2, e, \tilde{p}_1, \mathrm{ks})} H^2(X, \mathbb{Z}/2) \times H^4(X, \mathbb{Z}) \times H^4(X, \mathbb{Z}) \times H^4(X, \mathbb{Z}/2),$$

$$\mathrm{Mic}_m^+(X) \xrightarrow{(w_2, w_4, \tilde{p}_1, \mathrm{ks})} H^2(X, \mathbb{Z}/2) \times H^4(X, \mathbb{Z}/2) \times H^4(X, \mathbb{Z}) \times H^4(X, \mathbb{Z}/2), \text{ for } m > 4;$$

are injective and their images consist of the following sets of tuples respectively:

$$\{(a, c) : \rho_4 c = Pa\},$$

$$\{(a, b, c, d) : \rho_4 c = Pa + 2\rho_4 b\},$$

$$\{(a, b, c, d) : \rho_4 c = Pa + \iota_* b\},$$

where $P : H^2(-, \mathbb{Z}/2) \to H^4(-, \mathbb{Z}/4)$ denotes the Pontryagin square, $\iota_* : H^4(-, \mathbb{Z}/2) \to H^4(-, \mathbb{Z}/4)$ denotes the multiplication by 2, and $\mathrm{Mic}_m^+(X)$ denotes isomorphism classes of oriented m-microbundles.

As we mentioned, we know a few homotopy groups of $\mathrm{Sm}^{+,sf}(M)$; therefore, Galvin's result on the level of π_0 (Theorem 1.6) leads to the following natural conjecture which we could not prove or disprove so far.

Conjecture 1.7. *Let M be a connected, orientable, closed, smooth 4-manifold. Then the map*

$$\pi_k \operatorname{Homeo}(M \# S^2 \times S^2) \xrightarrow{\alpha_*} \pi_k \operatorname{Sm}^{+,sf}(M \# S^2 \times S^2) \simeq H^{3-k}(M \# S^2 \times S^2, \mathbb{Z}/2)$$

is surjective for $k = 1, 2$.

1.2 Overview of thesis organization

This thesis is structured in two parts: the first part consists of sections 2 through 5 in which we gather known results about microbundles, spherical fibrations, smooth structures, obstruction theory, and characteristic classes which we need to prove our main results. The second part consists of sections 6 and 7 in which we patiently perform computations using tools introduced in the first part to prove theorems A and B.

We begin in Section 2 by introducing microbundles and spherical fibrations. We also introduce their classifying spaces $\mathrm{BTop}(m)$, $\mathrm{BG}(m)$ which play an important role in the later sections.

Section 3 provides an introduction to smoothing theory which studies smooth structures on a topological manifold and crucially uses microbundles. In this section, we give a short outline of Kirby and Siebenmann's result on the classification of smooth structures and apply it to compute homotopy groups of Top/O which we need for our main result.

In Section 4 we introduce basic concepts of obstruction theory such as the Postnikov towers and k-invariants. These notions are essential for Theorem A. We prove some standard facts about these objects and, most importantly, Lemma 4.12 which will be our main computational tool in Section 6.

We continue with a short Section 5 in which we recall definitions of characteristic classes for vector bundles, microbundles, and spherical fibrations. We recall Wu's formula, which determines how the Steenrod algebra acts on the Stiefel-Whitney classes and emphasize that this formula also holds for spherical fibrations. Moreover, we introduce a characteristic class specific to microbundles – the Kirby-Siebenmann class, which plays an important role in our classification of microbundles (Theorem B).

In Section 6 we compute low dimensional homotopy groups of $\mathrm{BSO}(4)$, $\mathrm{BSTop}(4)$, $\mathrm{BSG}(4)$ and their Postnikov towers through dimension 5. We extensively rely on results from the previous sections to carry out these computations. In the end, this computation helps us to conclude Theorem A.

In the final Section 7 we recall the theorem of Dold and Whitney on the classification of vector bundles and use Theorem A to prove an analogous classification of microbundles – Theorem B. We avoid using the theorem of Dold and Whitney in our argument and perform a computation of Moore-Postnikov towers, which is slightly different from Dold and Whitney's original argument.

2 Vector bundles, Microbundles and Spherical Fibrations

In this section, we recall standard information about vector bundles, microbundles, spherical fibrations, and introduce some common notation which will be used later throughout the thesis. Smooth manifolds admit tangent vector bundles and microbundles are supposed to model this notion for topological manifolds. Milnor introduced them in [Mil64] to study smooth structures on manifolds. Similarly, spherical fibrations model an analogue of normal bundles of Poincaré duality spaces and are very important for surgery theory.

Definition 2.1. An m-dimensional **microbundle** over a space B is a triple (E, i, p), where $i : B \longrightarrow E$ and $p : E \longrightarrow B$ are maps which satisfy:

1. $p \circ i = \mathrm{Id}_B$,
2. For every $b \in B$ there are neighborhoods $U \subset B$, $V \subset E$ of b and $i(b)$ such that $i(U) \subset V$, $p(V) \subset U$, and there is a homeomorphism $h : V \longrightarrow U \times \mathbb{R}^m$ which makes the following diagram commutative

$$\begin{array}{ccccc} & & V & & \\ & \overset{i}{\nearrow} & \Big\downarrow h & \overset{p}{\searrow} & \\ U & & & & U \\ & \underset{\times 0}{\searrow} & & \underset{p_1}{\nearrow} & \\ & & U \times \mathbb{R}^m. & & \end{array}$$

The map i can be thought of as the zero section. Notice that the local triviality condition is very similar to the one of vector bundles; however, we only ask for a trivialization in a neighborhood of the zero section. Isomorphisms of microbundles are also supposed to capture this local behaviour near the zero section.

Definition 2.2. Let $B \xrightarrow{i_k} E_k \xrightarrow{p_k} B, k = 1, 2$ be two m-microbundles over the same base space, then an isomorphism between them is a homeomorphism $h : V_1 \longrightarrow V_2$ such that V_k are neighborhoods of $i_k(B)$ and the following diagram commutes

$$\begin{array}{ccccc} & & V_1 & & \\ & \overset{i_1}{\nearrow} & \Big\downarrow h & \overset{p_1}{\searrow} & \\ B & & & & B \\ & \underset{i_2}{\searrow} & & \underset{p_2}{\nearrow} & \\ & & V_2. & & \end{array}$$

Here are a few examples of microbundles:

Example 2.3. Let $p : E \longrightarrow B$ be a vector bundle and $s : B \longrightarrow E$ the zero section, then

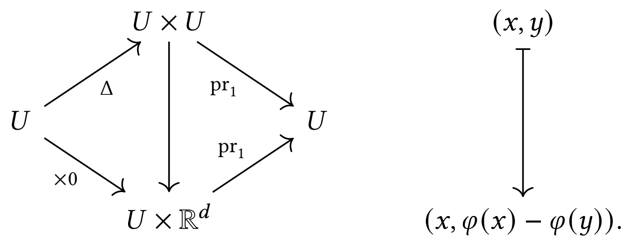
$$B \xrightarrow{s} E \xrightarrow{p} B$$

is a microbundle.

Example 2.4. Let M be a topological manifold and $\Delta : M \longrightarrow M \times M$ is the diagonal, then

$$M \xrightarrow{\Delta} M \times M \xrightarrow{\mathrm{pr}_1} M$$

is a microbundle. Indeed, $\mathrm{pr}_1 \circ \Delta = \mathrm{Id}_M$ and for any $x \in M$ there is a chart $\varphi : U \longrightarrow \mathbb{R}^d$ which gives us a local trivialization



This microbundle is called the **tangent microbundle** of M and is denoted by $\mathfrak{t}M$.

Example 2.5. Let $f : X \longrightarrow Y$ be a map between two spaces and $\xi = (E, i, p)$ a microbundle over Y. We define the pullback $f^*\xi := (E', i', p')$ as usual, then local triviality follows immediately.

The definition of the tangent microbundle is coherent with the notion of the tangent vector bundle of a smooth manifold.

Proposition 2.6 ([Mil64]). *Let M be a smooth manifold, then the underlying microbundle of the tangent vector bundle TM is isomorphic to $\mathfrak{t}M$.*

It is quite natural to expect that the tangent space of a topological manifold should be a fiber bundle with the fiber $\mathbb{R}^d$ (without the vector space structure). From the first view, the definition of a microbundle does not seem to be connected in any way to $\mathbb{R}^d$-bundles. However, a deep result of Kister and Mazur shows that if the base space is good enough then microbundles are actually the same as $\mathbb{R}^d$-bundles.

Theorem 2.7 ([Kis64]). *Let B be a locally finite simplicial complex or a topological manifold and let $\xi = (E, i, p)$ be an m-dimensional microbundle over B. Then E contains a unique, up to isomorphism, fiber bundle over B with fiber $\mathbb{R}^m$ and structure group $\mathrm{Top}(m) := \mathrm{Homeo}_0(\mathbb{R}^m)$ (homeomorphisms preserving the origin).*

This fact helps to study microbundles using homotopy theory because fiber bundles are better understood from this point of view. In the next section we will see how microbundles help to classify smooth structures on manifolds. Now we also introduce spherical fibrations.

Definition 2.8.

1. A Hurewicz fibration $p : E \longrightarrow B$ is called an m-dimensional **spherical fibration** if every fiber is homotopy equivalent to S^m.

2. An isomorphism between two spherical fibrations over the same base $E_k \xrightarrow{p_k} B, k = 1, 2$ is a fiber homotopy equivalence $E_1 \longrightarrow E_2$ over B.

One easy source of examples of spherical fibrations is complements of the zero section in the underlying $\mathbb{R}^m$-bundles of microbundles (or in vector bundles).
Let $\mathrm{Bun}_G(X)$ denote the functor which associates isomorphism classes of principal G-bundles to a space X. Recall the classification theorem for principal G-bundles.

Theorem 2.9. *([Die08]) There is a natural isomorphism of contravariant functors*

$$[-, \mathrm{BG}] \to \mathrm{Bun}_G(-)$$

from the homotopy category of CW complexes to the category of sets given by pulling back the universal principal G-bundle $\mathrm{EG} \to \mathrm{BG}$.

As a corollary of this theorem, we can derive a classification theorem for vector bundles. Let $\mathrm{Vect}_m(X)$ denote the functor which associates isomorphism classes of real m-dimensional vector bundles to a space X. Then we have

Corollary 2.10. *There is a natural isomorphism of contravariant functors* $\mathrm{Vect}_m(-)$ *and* $[-, \mathrm{BO}(m)]$ *from the homotopy category of CW complexes to the category of sets.*

Proof. Firstly, notice that there are natural inverse bijections

$$\mathrm{Bun}_{\mathrm{GL}_m(\mathbb{R})}(X) \to \mathrm{Vect}_m(X)$$

$$E \mapsto E \times_{\mathrm{GL}_m(\mathbb{R})} \mathbb{R}^m$$

where $E \times_{\mathrm{GL}_m(\mathbb{R})} \mathbb{R}^m$ denotes the quotient of $E \times \mathbb{R}^m$ by the equivalence relation

$$(x, y) \sim (x', y') \iff \exists g \in \mathrm{GL}_m(\mathbb{R}) : (x, y) = (x'g, g^{-1}y'),$$

and

$$\mathrm{Vect}_m(X) \to \mathrm{Bun}_{\mathrm{GL}_m(\mathbb{R})}(X)$$

$$E \mapsto \mathrm{Fr}(E)$$

where $\mathrm{Fr}(E)$ denotes the frame bundle of E (as a set, the total space of the frame bundle is $\bigsqcup_{x \in X} \mathrm{Fr}(F_x)$, where $\mathrm{Fr}(F_x)$ is the set of frames in the fiber over a point $x \in X$). Secondly, there is a deformation retraction $\mathrm{GL}_m(\mathbb{R}) \simeq \mathrm{O}(m)$; therefore, by Theorem 2.9 we have isomorphisms of functors

$$\mathrm{Vect}_m(-) \simeq \mathrm{Bun}_{\mathrm{GL}_m(\mathbb{R})}(-) \simeq [-, \mathrm{BGL}_m(\mathbb{R})] \simeq [-, \mathrm{BO}(m)].$$

□

Let $\mathrm{Mic}_m(X)$ denote the functor which associates isomorphism classes of m-dimensional microbundles to a space X. Then a similar argument combined with Theorem 2.7 implies a classification theorem for microbundles.

Corollary 2.11. *There is a natural isomorphism of contravariant functors* $\mathrm{Mic}_m(-)$ *and* $[-, \mathrm{BTop}(m)]$ *from the category of topological manifolds or locally finite simplicial complexes and homotopy classes of continuous maps to the category of sets.*

Remark 2.12. Note that a proof would use Theorem 2.9, which is applicable in this context because any topological manifold has homotopy type of a CW complex. Moreover, one also has to check that $\mathrm{Mic}_m(-)$ defines a functor on the category of manifolds (or locally finite simplicial complexes) and homotopy classes of continuous maps. For this, it is enough to show that if $f, g : X \to Y$ are homotopic maps and ξ is a microbundle over Y then there is an isomorphism $f^*\xi \simeq g^*\xi$. This is indeed true for paracompact spaces by Theorem 3.1 in [Mil64], we will also call it the **microbundle homotopy theorem**.

Moreover, Stasheff proved that an analogous classification theorem also holds for spherical fibrations. Let $\mathrm{Sph}_m(X)$ denote the functor which associates isomorphism classes of m-dimensional spherical fibrations to a space X and $\mathrm{G}(m) := \mathrm{hAut}(S^{m-1})$ the monoid of self-homotopy equivalences of S^{m-1}, then we have

Theorem 2.13. *([Sta63]) There is a natural isomorphism of contravariant functors* $\mathrm{Sph}_{m-1}(-)$ *and* $[-, \mathrm{BG}(m)]$ *from the homotopy category of CW complexes to the category of sets.*

Now we introduce more notation which will be used later. We write $\mathrm{STop}(m)$ and $\mathrm{SG}(m)$ for the spaces of orientation preserving homeomorphisms of $\mathbb{R}^m$ and self-homotopy equivalences of S^{m-1} respectively. Write $\mathrm{F}(m) := \mathrm{hAut}_*(S^m)$ and $\mathrm{SF}(m) := \mathrm{hAut}_*^+(S^m)$ for the spaces of pointed homotopy equivalences of S^m and its oriented version respectively. Finally, we also introduce the stable versions of the classifying spaces

$$\mathrm{BTop} := \mathrm{hocolim}_m \, \mathrm{BTop}(m)$$

where the colimit is taken over the maps induced by inclusions $\mathrm{Top}(m) \to \mathrm{Top}(m+1)$;

$$\mathrm{BG} := \mathrm{hocolim}_m \, \mathrm{BG}(m)$$

where the colimit is taken over the maps induced by

$$\mathrm{G}(m) \to \mathrm{G}(m+1)$$

$$f \mapsto \Sigma f.$$

We conclude this section by defining the notion of orientability for microbundles and spherical fibrations. Let $\xi = (E, i, p)$ be an m-microbundle over B, let $b \in B$, and let $V \subset E$ be a locally trivial neighborhood of $i(b)$; then

$$H^m(p^{-1}(b), p^{-1}(b) - i(B); \mathbb{Z}) \simeq H^m(p^{-1}(b) \cap V, (p^{-1}(b) - i(B)) \cap V; \mathbb{Z}) \simeq H^m(\mathbb{R}^m, \mathbb{R}^m - 0; \mathbb{Z}) \simeq \mathbb{Z}.$$

We say that ξ is **orientable** if there is a class $u \in H^m(E, E - i(B); \mathbb{Z})$ such that its restriction to $H^m(p^{-1}(b), p^{-1}(b) - i(B); \mathbb{Z})$ is a generator for all $b \in B$; then u is called a **Thom class** of ξ. Now let $\xi = (E \to B)$ be an $(m-1)$-dimensional spherical fibration, $p' : \mathrm{Cyl}(p) \to B$ and $b \in B$; then again

$$H^m(p'^{-1}(b), p'^{-1}(b) - b; \mathbb{Z}) \simeq H^m(D^m, D^m - 0; \mathbb{Z}) \simeq \mathbb{Z}.$$

We say that ξ is **orientable** if there is a class $u \in H^m(\mathrm{Cyl}(p), \mathrm{Cyl}(p) - B; \mathbb{Z})$ such that its restriction to $H^m(p'^{-1}(b), p'^{-1}(b) - b; \mathbb{Z})$ is a generator for all $b \in B$; then u is called a **Thom**

class of ξ.
We also say that a vector bundle, a microbundle, or a spherical fibration is **oriented** if it is orientable and we fixed a choice of a Thom class u. Being orientable is a property, whereas being oriented is an additional data.

Remark 2.14. The same classification theorems also hold for oriented vector bundles, microbundles, and spherical fibrations if we also replace the classifying spaces with their oriented versions. The proof is very similar, but one also has to argue that being orientable is equivalent to the vanishing of w_1 (see Section 5 for more details).

3 Smoothing theory

In this section, we reconstruct some of the fundamental results in the smoothing theory of Kirby and Siebenmann. Our exposition mainly follows [KS77].
Kirby and Siebenmann developed a very extensive machinery which allows one to work with topological manifolds of dimension at least 5. Here is a small, non-exhaustive list of their striking results: topological transversality theorem, topological isotopy extension theorem, existence of topological handle decompositions, and topological surgery. Our main goal in this section is to give an outline of the computation of homotopy groups of Top/O (Theorem 3.18) which we will use in Section 6. The main ingredient for this computation is the theorem of Kirby and Siebenmann on the classification of smooth structures on a topological manifold (3.14), therefore we mainly focus on this theorem.

3.1 Spaces of smooth structures

In this section we mainly will be working with simplicial sets instead of topological spaces as they provide a more convenient framework. Given a topological manifold M without boundary we write $\mathrm{Homeo}(M)$ for the simplicial group with a typical k-simplex given by a commutative diagram

$$\begin{array}{ccc} M \times \Delta^k & \xrightarrow{\quad h \quad} & M \times \Delta^k \\ & {\scriptstyle \mathrm{pr}_2} \searrow \quad \swarrow {\scriptstyle \mathrm{pr}_2} & \\ & \Delta^k & \end{array}$$

with h being a homeomorphism, note that it is the same as the singular simplicial set of the topological space version of the homeomorphism group $\mathrm{Sing}(\mathrm{Homeo}(M))$. Similarly, given a smooth manifold M without boundary, we write $\mathrm{Diff}(M)$ for the simplicial group with a typical k-simplex given by a commutative diagram

$$\begin{array}{ccc} M \times \Delta^k & \xrightarrow{\quad h \quad} & M \times \Delta^k \\ & {\scriptstyle \mathrm{pr}_2} \searrow \quad \swarrow {\scriptstyle \mathrm{pr}_2} & \\ & \Delta^k & \end{array}$$

with h being a diffeomorphism of manifolds with corners. This simplicial group is not isomorphic to $\mathrm{Sing}(\mathrm{Diff}(M))$ but, using Whitney's approximation theorem, one can show that they are weakly equivalent.
We will make frequent use of the following bundle theorem of Kirby and Siebenmann.

Theorem 3.1 (Bundle theorem, [KS77] 0.1 on page 217)**.** *Let M be a topological manifold with $\dim M \neq 4 \neq \dim \partial M$ and let Σ be a smooth structure on $\Delta^k \times M$ such that the projection $p_2 : (M \times \Delta^k)_\Sigma \to \Delta^k$ is a smooth submersion onto the standard smooth k-simplex (with corners, $k \geq 0$). Then there is a smooth structure γ on M and a diffeomorphism $h : (M \times \Delta^k)_\Sigma \to M_\gamma \times \Delta^k$*

such that the following diagram commutes

$$\begin{array}{ccc} (M \times \Delta^k)_\Sigma & \xrightarrow[h]{\hspace{3em}} & M_\gamma \times \Delta^k \\ & {\scriptstyle p_2}\searrow \quad \swarrow{\scriptstyle p_2} & \\ & \Delta^k. & \end{array}$$

We do not provide a proof, the general idea is to cut M into compact pieces for which the statement follows from Ehresmann's lemma, and then patch them together. This process of cutting M into pieces is in fact quite complicated and uses engulfing. Now we define the space of smooth structures which will be our main object of interest for the rest of the section.

Definition 3.2. For a topological manifold M without boundary we write $\mathrm{Sm}(M)$ for the simplicial set with a typical k-simplex given by a smooth structure on $M \times \Delta^k$ (which we denote as $(M \times \Delta^k)_\Sigma$) such that the projection map

$$\mathrm{pr}_2 : (M \times \Delta^k)_\Sigma \to \Delta^k$$

is a smooth submersion. Given a map $\lambda : [k] \to [l]$, we define an induced map

$$\mathrm{Sm}(M)_l \to \mathrm{Sm}(M)_k$$

by pulling back bundles $(M \times \Delta^l)_\Sigma \to \Delta^l$ along the map $\lambda_* : \Delta^k \to \Delta^l$. These pullbacks are smooth and the projection

$$(\lambda_*)^*(M \times \Delta^l)_\Sigma \to \Delta^k$$

is a submersion because the map

$$\Delta^k \times (M \times \Delta^l)_\Sigma \xrightarrow{\lambda_* \times \mathrm{pr}_2} \Delta^l \times \Delta^l$$

is transverse to the diagonal.

Note that $\pi_0 \,\mathrm{Sm}(M) = \{\Sigma : \Sigma \text{ is a smooth structure on M}\}/\sim$, where $\Sigma \sim \Sigma'$ if and only if there exists a smooth structure $\overline{\Sigma}$ on $M \times I$ which restricts to Σ and Σ' on the endpoints and such that the projection map

$$\mathrm{pr}_2 : (M \times I)_{\overline{\Sigma}} \to I$$

is a smooth submersion. This equivalence relation is called **sliced concordance** of smooth structures (if we relax the condition of the projection onto I being a submersion, then the equivalence relation would be called just **concordance**). There is also another equivalence relation on smooth structures which is called isotopy. M_Σ is **isotopic** to $M_{\Sigma'}$ if there is a continuous family of homeomorphisms

$$h_t : M_\Sigma \times I \to M_{\Sigma'},$$

such that $h_0 = \mathrm{Id}$ and $h_1^*\Sigma' = \Sigma$. Isotopy implies diffeomorphism and sliced concordance of smooth structures; a priori the converse implications do not hold. However, the bundle theorem

implies that isotopy follows from sliced concordance in the appropriate dimensions.
The classification theorem 3.14 tells us that the tangential data fully captures information about smooth structures if the dimension is not 4. We introduce a couple of spaces which "interpolate" between smooth structures and tangential data.
Let M be a smooth manifold without boundary and $\xi = (E, i, p)$ a microbundle over M, an **almost smooth microbundle structure** Σ on ξ is a smooth structure on an open neighborhood U of the zero section $i(M)$ such that $U_\Sigma \xrightarrow{p} M$ is a smooth submersion. We also say that Σ is a **smooth microbundle structure** on ξ if, in addition, $i : M \to U_\Sigma$ is smooth. Two smooth structures Σ, Σ' on ξ have the same **germ about the zero section** if there is an open neighborhood of $i(M)$ on which $\Sigma = \Sigma'$.

Definition 3.3. Let M be a smooth manifold without boundary and let ξ be a microbundle over M, then

1. $\mathrm{aSm}(\xi)$ denotes the simplicial set with a typical k-simplex given by a germ of **almost smooth microbundle structures** on the product microbundle $\xi \times \Delta^k$ over $M \times \Delta^k$,
2. $\mathrm{Sm}(\xi)$ denotes the simplicial set with a typical k-simplex given by a germ of **smooth microbundle structures** on the product microbundle $\xi \times \Delta^k$ over $M \times \Delta^k$.

It is clear that $\mathrm{Sm}(\xi)$ embeds into $\mathrm{aSm}(\xi)$, this embedding is usually a homotopy equivalence, as we will see later. For a smooth manifold M we define a map

$$d : \mathrm{Sm}(M) \to \mathrm{aSm}(\mathfrak{t}M)$$

as follows, for $\Sigma \in \mathrm{Sm}(M)_k$ set $d\Sigma$ to be the germ of the smooth structure $M \times \Sigma$ on $E(\mathfrak{t}M \times \Delta^k) = M \times M \times \Delta^k$.

Remark 3.4.

1. The image of d does not land in $\mathrm{Sm}(\mathfrak{t}M)$ because $\mathrm{Id} : M \to M_\Sigma$ is almost never smooth,
2. It is possible to define $d : \mathrm{Sm}(M) \to \mathrm{aSm}(\hat{\mathfrak{t}}M)$ more generally without assuming smoothness of M. This is useful for stating all the upcoming theorems in the greatest generality. Let $M \to N$ be a topological embedding into a smooth manifold N with a continuous retraction $r : N \to M$ (all manifolds are ENRs, so such a pair always exists). Consider the pullback $\hat{\mathfrak{t}} = r^*\mathfrak{t}M$ over N, or more explicitly $\hat{\mathfrak{t}} = (M \times N, i, \mathrm{pr}_2)$ where $i(y) = (r(y), y)$. N is smooth and therefore $\mathrm{aSm}(\hat{\mathfrak{t}})$ is well-defined and we have a map

   $$\mathrm{Sm}(M) \to \mathrm{aSm}(\hat{\mathfrak{t}})$$
   $$\Sigma \mapsto \Sigma \times N.$$

   We define

   $$\mathrm{aSm}(\hat{\mathfrak{t}}M) := \mathrm{colim}_U \, \mathrm{aSm}(\hat{\mathfrak{t}}|_U),$$

   where the colimit is taken over the directed system $\{U : U \text{ is open in } N \text{ and } M \subset U\}$, then we also have the restriction map $\mathrm{aSm}(\hat{\mathfrak{t}}) \to \mathrm{aSm}(\hat{\mathfrak{t}}M)$. We define

   $$d : \mathrm{Sm}(M) \to \mathrm{aSm}(\hat{\mathfrak{t}}M)$$

as the composition of the two maps above. If M is smooth and $N = M$ we have the same d as we defined before.

Now we prove that in the dimensions for which the bundle theorem is valid, simplicial sets $\mathrm{Sm}(M), \mathrm{Sm}(\xi), \mathrm{aSm}(\xi)$ are Kan complexes and therefore they deserve to be called "spaces".

Proposition 3.5 ([KS77] page 229)**.** *Let M be a topological manifold without boundary and $\dim M \neq 4$, then $\mathrm{Sm}(M)$ is a Kan complex.*

Proof. Assume $\mathrm{Sm}(M)$ is not empty (otherwise the statement is trivial), let Σ be a smooth structure on M. $\mathrm{Diff}(M_\Sigma)$ acts freely on the left on $\mathrm{Homeo}(M)$, then by 17.1, 18.2 in [May68] $\mathrm{Diff}(M_\Sigma) \to \mathrm{Homeo}(M) \to \mathrm{Homeo}(M)/\mathrm{Diff}(M_\Sigma)$ is a Kan fibration and $\mathrm{Homeo}(M)/\mathrm{Diff}(M_\Sigma)$ is a Kan complex. Moreover, we have an injective map of simplicial sets

$$\Sigma_* : \mathrm{Homeo}(M)/\mathrm{Diff}(M_\Sigma) \to \mathrm{Sm}(M)$$

$$h \mapsto h(\Sigma \times \Delta^k).$$

The bundle theorem shows that Σ_* is also surjective onto the connected components which it hits. Now we can vary Σ so the images of Σ_* cover the whole $\mathrm{Sm}(M)$ and we are done. □

Remark 3.6. $\mathrm{Sm}(M)$ fails to be a Kan complex in dimension 4. Therefore, it is reasonable to define the space of smooth structures differently for 4-manifolds. The proof of the previous proposition suggests that one can define it as the homotopy fiber

$$\mathrm{fib}\Big(\bigsqcup_{\Sigma} \mathrm{BDiff}(M_\Sigma) \to \mathrm{BHomeo}(M)\Big),$$

where the disjoint union is taken over the diffeomorphism classes of the smooth structures on M. We will not use this definition later as none of the results in this section apply to 4-manifolds, but we include it for the completeness of the exposition.

A similar argument also shows that $\mathrm{aSm}(\xi)$ is a Kan complex.

Proposition 3.7 ([KS77] page 229)**.** *Let M be a smooth manifold without boundary and let ξ be a microbundle of dimension $m \neq 4$, then $\mathrm{aSm}(\xi)$ is a Kan complex.*

Proof. Let $(E(\xi) \times \Delta^k)_\Sigma$ be a k-simplex in $\mathrm{aSm}(\xi)$. By the Kister-Mazur theorem we can assume that $\xi \times \Delta^k$ is a locally trivial $\mathbb{R}^m$ bundle since we are working with germs of smooth structures on $E(\xi) \times \Delta^k$. Moreover, by the bundle theorem $(E(\xi) \times \Delta^k)_\Sigma \to M \times \Delta^k$ is a locally trivial smooth fiber bundle. By the homotopy theorem for fiber bundles, there is an isomorphism

$$(E(\xi) \times \Delta^k)_\Sigma \cong E(\xi)_\gamma \times \Delta^k$$

of smooth $\mathrm{Diff}(\mathbb{R}^m)$ fiber bundles over $M \times \Delta^k$ where γ is the restriction of Σ to $E(\xi) \times 0$ (one has to be careful with corners here, we come back to this issue after finishing the argument). Now, similarly to Proposition 3.5, we have a Kan fibration

$$\mathrm{Diff}(\xi') \to \mathrm{Homeo}(\xi) \to \mathrm{Homeo}(\xi)/\mathrm{Diff}(\xi'),$$

where ξ' is a zero simplex in $\mathrm{aSm}(\xi)$, again there is an injective map $\mathrm{Homeo}(\xi)/\mathrm{Diff}(\xi') \to \mathrm{aSm}(\xi)$ and the argument above shows that it is surjective onto the path components which it hits. By varying ξ' we conclude that $\mathrm{aSm}(\xi)$ is a Kan complex.
We come back to the issue with corners, let $r_t : I \times \Delta^k \to \Delta^k$ be a smooth homotopy from identity to a map which preserves $\partial\Delta^k$ and maps an open neighborhood of the boundary to the boundary. One can construct such a homotopy by composing $k+1$ homotopies, each of which pushes a neighborhood of a codimension 1 face onto the face itself, radially from the opposite vertex (this homotopy does not fix $\partial\Delta^k$ pointwise, but preserves it as a set). Define the map

$$\rho : I \times M \times \Delta^k \to I \times M \times \Delta^k$$

$$(t, x, y) \mapsto (t, x, r_t(y)),$$

then $\rho^*(I \times (E(\xi) \times \Delta^k)_\Sigma)$ gives a smooth structure Γ on $I \times E(\xi) \times \Delta^k$ as a bundle over $I \times M \times \Delta^k$ such that

$$\Gamma|_{0\times M\times\Delta^k} = (E(\xi) \times \Delta^k)_\Sigma$$

and $\Gamma|_{1\times M\times\Delta^k}$ extends to a smooth structure Σ' on the bundle $E(\xi) \times \mathbb{R}^{k+1}$ over $M \times \mathbb{R}^{k+1}$. Now by the homotopy theorem for fiber bundles

$$(E(\xi) \times \Delta^k)_\Sigma \cong (E(\xi) \times \Delta^k)_{\Gamma_1}$$

and

$$(E(\xi) \times \Delta^k)_{\Gamma_1} \cong E(\xi)_\gamma \times \Delta^k,$$

because we can again apply the homotopy theorem for fiber bundles $k+1$ times to $(E(\xi) \times I^{k+1})_{\Sigma'}$, thus we are done. □

A proof of the fact that $\mathrm{Sm}(\xi)$ is a Kan complex is even easier, one replaces the use of the bundle theorem with the more common tubular neighborhood theorem; moreover, the argument works in all dimensions.
The most difficult part of proving the classification theorem 3.14 is the following

Proposition 3.8 ([KS77] 1.4 on page 222)**.** *Let M be a topological manifold without boundary and $\dim M \neq 4$ then the map $d : \mathrm{Sm}(M) \to \mathrm{aSm}(\hat{\mathfrak{t}}M)$ is a weak equivalence.*

We will not provide a full argument but give a short outline of the steps which Kirby and Siebenmann do:

1. $\mathrm{Sm}(U)$ and $\mathrm{aSm}(\hat{\mathfrak{t}}U)$ are contravariant functors on open subsets in M and they satisfy the sheaf condition. In other words, the diagrams

$$\mathrm{Sm}(U) \longrightarrow \textstyle\prod_i \mathrm{Sm}(U_i) \rightrightarrows \textstyle\prod_{i,j} \mathrm{Sm}(U_{ij}),$$

$$\mathrm{aSm}(\hat{\mathfrak{t}}U) \longrightarrow \textstyle\prod_i \mathrm{aSm}(\hat{\mathfrak{t}}U_i) \rightrightarrows \textstyle\prod_{i,j} \mathrm{aSm}(\hat{\mathfrak{t}}U_{ij})$$

are equalizers for any open cover $\{U_i\}_i$ of U. For any subset $A \subset M$ define

$$\mathrm{Sm}_M(A) := \mathrm{colim}_U \mathrm{Sm}(U),$$

$$\mathrm{aSm}_{\hat{\mathfrak{t}}M}(A) := \mathrm{colim}_U \, \mathrm{aSm}(\hat{\mathfrak{t}}U),$$

where colimits are taken over the directed system $\{U : A \subset U$ and U is open in $M\}$. Note that we still have a map

$$d_A : \mathrm{Sm}_M(A) \to \mathrm{aSm}_{\hat{\mathfrak{t}}M}(A).$$

2. Let B be a smooth simplex in a chart of M. Then the restriction maps

$$\mathrm{Sm}_M(B) \to \mathrm{Sm}_M(*),$$

$$\mathrm{aSm}_{\hat{\mathfrak{t}}M}(B) \to \mathrm{aSm}_{\hat{\mathfrak{t}}M}(*)$$

are equivalences for any interior point of B.

3. Let x be a point in M. Then the map

$$d_x : \mathrm{Sm}_M(x) \to \mathrm{aSm}_{\hat{\mathfrak{t}}M}(x)$$

is a weak equivalence.

4. Let $A \subset B$ be a pair of subsets in M. Then the restriction maps

$$\mathrm{Sm}_M(B) \to \mathrm{Sm}_M(A),$$

$$\mathrm{aSm}_{\hat{\mathfrak{t}}M}(B) \to \mathrm{aSm}_{\hat{\mathfrak{t}}M}(A)$$

are Kan fibrations (this step uses a more general bundle theorem from the Essay 2 in [KS77]).

5. Given facts from steps 1-4 the proof reduces to the standard machinery of Gromov's h-principle. Fix a handle decomposition for M (all manifolds with $\dim M \neq 4$ have one), one can start with a homotopy equivalence from steps 2 and 3 on the zero handles and then proceed by induction on handles of higher index using steps 1 and 4.

3.2 Classification of smooth structures

So far we have reduced the question about understanding the homotopy type of the space $\mathrm{Sm}(M)$ to $\mathrm{aSm}(\hat{\mathfrak{t}}M)$. We make a couple more simplifications of this homotopy type before we prove the classification theorem 3.14.

Lemma 3.9 ([KS77] 2.1 on page 234)**.** *Let M be a smooth manifold without boundary and let ξ be a microbundle over M with $\dim \xi \neq 4$, then the inclusion map $\mathrm{Sm}(\xi) \to \mathrm{aSm}(\xi)$ is a weak equivalence.*

Proof. Firstly, notice that in this case, both spaces are Kan complexes by Proposition 3.7 and the paragraph after it, so it is sufficient to prove that for any map of pairs

$$\Sigma : (\Delta^k, \partial \Delta^k) \to (\mathrm{aSm}(\xi), \mathrm{Sm}(\xi))$$

there is a lift $\Delta^k \to \mathrm{Sm}(\xi)$ up to homotopy relative to $\partial\Delta^k$. The map Σ gives us a smooth structure $\Sigma \times I$ on $E(\xi) \times \Delta^k \times I$ such that

$$p : E(\xi) \times \Delta^k \times I \to M \times \Delta^k \times I$$

is a smooth submersion and

$$i : M \times \Delta^k \times I \to E(\xi) \times \Delta^k \times I$$

is smooth near $M \times \partial\Delta^k \times I$. Now we construct another microbundle ζ over $M \times \Delta^k \times I$ by slightly changing the map i. As we have seen in Proposition 3.7 we can assume that

$$p : E(\xi) \times \Delta^k \times I \to M \times \Delta^k \times I$$

is a smooth bundle with structure group $\mathrm{Diff}(\mathbb{R}^m)$ by a combination of the Kister-Mazur and bundle theorems. Now we apply the relative Whitney approximation theorem chart by chart in the fiber direction to construct another section i' of p which coincides with i on $M \times \partial\Delta^k \times I \cup M \times \Delta^k \times 1$ and is smooth on $M \times \Delta^k \times 0$. By the relative version of microbundle homotopy theorem, we have a homeomorphism of microbundles

$$H : \zeta \to \xi \times \Delta^k \times I$$

which is equal to the identity over $M \times \partial\Delta^k \times I \cup M \times \Delta^k \times 1$. The pushforward smooth structure $H_*(\Sigma \times I)$ is the same as $\Sigma \times I$ over $M \times \partial\Delta^k \times I \cup M \times \Delta^k \times 1$ and i is smooth on $M \times \Delta^k \times 0$ in this smooth structure. Therefore, $H_*(\Sigma \times I)$ provides the desired homotopy of Σ into $\mathrm{Sm}(\xi)$ relative to $\partial\Delta^k$. □

Now we introduce a few additional simplicial sets. For two topological spaces X and Y, define

$$\mathrm{Map}(X, Y) \coloneqq \mathrm{Sing}\,\mathrm{Map}^t(X, Y),$$

where $\mathrm{Map}^t(X, Y)$ is the space of maps endowed with the compact-open topology. Denote the universal microbundles over $\mathrm{BTop}(m)$ and $\mathrm{BO}(m)$ by γ_t^m and γ_s^m respectively.

Definition 3.10. Let M be a smooth manifold, $\mathrm{BSm}_m^{cls}(M)$ denotes the simplicial set with a typical k-simplex given by a smooth microbundle ξ over $M \times \Delta^k$ with a smooth microbundle map $\phi : \xi \to \gamma_s^m$. If M is a topological manifold we denote by $\mathrm{BTop}_m^{cls}(M)$ a similar simplicial set of topological microbundles with a classifying map. To avoid set-theoretical issues we ask microbundles to be subsets of $M \times \mathbb{R}^\infty$; two simplices are the identified if they coincide on an open neighborhood of the zero section.

Also, define simplicial sets $\mathrm{BSm}_m(M)$ and $\mathrm{BTop}_m(M)$ in the same way but omit the classifying maps.

Lemma 3.11 ([KS77] pages 236-238).

1. *Let M be a smooth manifold. Then the simplicial sets $\mathrm{BSm}_m^{cls}(M)$, $\mathrm{BSm}_m(M)$ are Kan complexes; the forgetful maps*

$$\mathrm{BSm}_m(M) \leftarrow \mathrm{BSm}_m^{cls}(M) \rightarrow \mathrm{Map}(M, \mathrm{BO}(m))$$

are weak equivalences; and the forgetful map

$$\mathrm{BSm}_m(M) \rightarrow \mathrm{BTop}_m(M)$$

is a Kan fibration.

2. *Let M be a topological manifold. Then the simplicial sets $\mathrm{BTop}_m^{cls}(M)$, $\mathrm{BTop}_m(M)$ are Kan complexes; and the forgetful maps*

$$\mathrm{BTop}_m(M) \leftarrow \mathrm{BTop}_m^{cls}(M) \rightarrow \mathrm{Map}(M, \mathrm{BTop}(m))$$

are weak equivalences.

We omit the proof, but the idea is very similar to Proposition 3.7 combined with the use of the universal property of bundles γ_t^m, γ_s^m.
We also write $\mathrm{Sm}^f(M)$ for the homotopy fiber $\mathrm{fib}_{\mathfrak{t}M}(\mathrm{Map}(M, \mathrm{BO}(d)) \rightarrow \mathrm{Map}(M, \mathrm{BTop}(d)))$ and call it the **space of formally smooth structures** on M (note that it is just the space of homotopy lifts of the tangent microbundle $M \rightarrow \mathrm{BTop}(d)$ to $\mathrm{BO}(d)$). Let M be a smooth manifold, consider the following diagram of forgetful maps and vertical homotopy fibers

$$\begin{array}{ccccc}
\mathrm{Sm}(\mathfrak{t}M) & \longleftarrow & \mathrm{Sm}^{cls}(M) & \longrightarrow & \mathrm{Sm}^f(M) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{BSm}_d(M) & \longleftarrow & \mathrm{BSm}_d^{cls}(M) & \longrightarrow & \mathrm{Map}(M, \mathrm{BO}(d)) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{BTop}_d(M) & \longleftarrow & \mathrm{BTop}_d^{cls}(M) & \longrightarrow & \mathrm{Map}(M, \mathrm{BTop}(d)),
\end{array}$$

where $\mathrm{Sm}^{cls}(M)$ is defined as the fiber

$$\mathrm{Sm}^{cls}(M) := \mathrm{fib}_{\mathfrak{t}M}(\mathrm{BSm}_d^{cls}(M) \rightarrow \mathrm{BTop}_d^{cls}(M)).$$

Then by Lemma 3.11 we have a zig-zag of natural weak equivalences

$$\mathrm{Sm}(\mathfrak{t}M) \leftarrow \mathrm{Sm}^{cls}(M) \rightarrow \mathrm{Sm}^f(M) \tag{3.12}$$

and by taking direct limits we also have a zig-zag of weak equivalences for any subset $A \subset M$

$$\mathrm{Sm}_{\mathfrak{t}M}(A) \leftarrow \mathrm{Sm}_M^{cls}(A) \rightarrow \mathrm{Sm}_M^f(A). \tag{3.13}$$

Now we are finally ready to prove the classification theorem:

Theorem 3.14 ([KS77] 2.3 on page 235)**.** *Let M be a topological manifold without boundary and $\dim M = d \neq 4$, fix a classifying map $\varphi : \mathfrak{t}M \to \gamma^d_{\mathrm{Top}}$ covering $f : M \to \mathrm{BTop}(d)$, then there is a homotopy equivalence*

$$\theta : \mathrm{Sm}(M) \to \mathrm{Sm}^f(M)$$

well-defined up to homotopy and for any subset $A \subset M$ there is a homotopy equivalence

$$\theta_A : \mathrm{Sm}(A) \to \mathrm{Sm}^f(A).$$

These equivalences are natural: for any subsets $B \subset A \subset M$ the diagram

$$\begin{array}{ccc} \mathrm{Sm}(A) & \xrightarrow{\theta_A} & \mathrm{Sm}^f(A) \\ \downarrow & & \downarrow \\ \mathrm{Sm}(B) & \xrightarrow[\theta_B]{} & \mathrm{Sm}^f(B) \end{array}$$

commutes up to homotopy.

Proof. Fix an embedding $\iota : M \to N$ of M into a smooth manifold N with a retraction $r : N \to M$. Extend φ to $\hat{\varphi} : \hat{\mathfrak{t}} = r^*\mathfrak{t}M \to \gamma^d_t$ covering $\hat{f} : N \to \mathrm{BTop}(d)$ via the compositions

$$\begin{array}{ccccc} r^*\mathfrak{t}M & \longrightarrow & \mathfrak{t}M & \xrightarrow{\varphi} & \gamma^d_t \\ \downarrow & & \downarrow & & \downarrow \\ N & \xrightarrow{r} & M & \xrightarrow{f} & \mathrm{BTop}(d), \end{array}$$

then for any $A \subset M$ there is a homotopy equivalence

$$\mathrm{Lift}(\hat{f} \text{ to } \mathrm{BO}(d) \text{ near } A) \xrightarrow{\iota^*} \mathrm{Sm}^f_M(A),$$

where in the definition of $\mathrm{Sm}^f(M)$ we take the fiber over f and $\mathrm{Lift}(\hat{f} \text{ to } \mathrm{BO}(d) \text{ near } A)$ denotes the colimit of the spaces of lifts over the open neighborhoods of A. The homotopy inverse is induced by r, both M and N are ANRs and therefore there is a deformation retraction of a neighborhood of M inside N onto M, it induces a homotopy $\iota^* \circ r^* \simeq \mathrm{Id}$. Thus, by 3.8, 3.9, 3.11 we have a zig-zag of natural weak equivalences

$$\mathrm{Sm}_M(A) \xrightarrow{d} \mathrm{aSm}_{\hat{\mathfrak{t}}M}(A) = \mathrm{aSm}_{\hat{\mathfrak{t}}}(A) \leftarrow \mathrm{Sm}_{\hat{\mathfrak{t}}}(A) \leftarrow \mathrm{Sm}^{cls}_{\hat{\mathfrak{t}}}(A) \to$$

$$\to \mathrm{Lift}(\hat{f} \text{ to } \mathrm{BO}(d) \text{ near } A) \xrightarrow{\iota^*} \mathrm{Sm}^f_M(A),$$

and we can find the promised θ_A by taking homotopy inverses in these zig-zags. □

Remark 3.15. In the classification theorem, we chose an embedding of M into a smooth manifold. One can show that the resulting maps θ_A do not depend on this embedding up to homotopy (see [KS77] page 238).

With a bit more work, it is also possible to prove a version of the classification theorem when a manifold has boundary and when we fix a prescribed smooth structure on some subset. Let M be a topological manifold (possibly with boundary), let A be a subset of M, and let Σ_0 be a smooth structure on an open neighborhood U of A in M. Recall that the restriction map $\mathrm{Sm}(M) \xrightarrow{p} \mathrm{Sm}_M(A)$ is a Kan fibration (Proposition 3.8) and write

$$\mathrm{Sm}(M \text{ rel } A; \Sigma_0) := \mathrm{fib}_{\Sigma_0}(\mathrm{Sm}(M) \xrightarrow{p} \mathrm{Sm}_M(A)).$$

Also, let

$$f : M \to \mathrm{BTop}(d),$$
$$\partial f : \partial M \to \mathrm{BTop}(d-1)$$

be classifying maps of the tangent microbundles of M,∂M; and let

$$f_0 : U \to \mathrm{BO}(d)$$

$$\partial f_0 : \partial U \to \mathrm{BO}(d-1)$$

be lifts of $f, \partial f$ such that the following diagram

$$\begin{array}{ccc} \partial U & \xrightarrow[\partial f_0]{} & \mathrm{BO}(d-1) \\ \downarrow & & \downarrow \\ U & \xrightarrow[f_0]{} & \mathrm{BO}(d) \end{array}$$

commutes up to homotopy. Then define the space $\mathrm{Lift}(f, \partial f \text{ to } \mathrm{BO}(d), \mathrm{BO}(d-1))$ as the homotopy pullback

$$\begin{array}{ccc} \mathrm{Lift}(f, \partial f \text{ to } \mathrm{BO}(d), \mathrm{BO}(d-1)) & \longrightarrow & \mathrm{Lift}(f : M \to \mathrm{BTop}(d) \text{ to } \mathrm{BO}(d)) \\ \downarrow & & \downarrow{\scriptstyle \tilde{f} \mapsto \tilde{f}|_{\partial M}} \\ \mathrm{Lift}(\partial f : \partial M \to \mathrm{BTop}(d-1) \text{ to } \mathrm{BO}(d-1)) & \longrightarrow & \mathrm{Lift}(g : \partial M \to \mathrm{BTop}(d) \text{ to } \mathrm{BO}(d)), \end{array}$$

where the bottom horizontal map sends a lift to its postcomposition with the stabilization map. Also define the space $\mathrm{Lift}(f, \partial f \text{ to } \mathrm{BO}(d), \mathrm{BO}(d-1) \text{ near } A)$ as the colimit of the spaces $\mathrm{Lift}(f, \partial f \text{ to } \mathrm{BO}(d), \mathrm{BO}(d-1))$ taken over open neighborhoods of A in M; and denote

$$\mathrm{Lift}(f, \partial f \text{ to } \mathrm{BO}(d), \mathrm{BO}(d-1) \text{ rel } f_0, \partial f_0) :=$$

$$\mathrm{fib}_{f_0, \partial f_0}(\mathrm{Lift}(f, \partial f \text{ to } \mathrm{BO}(d), \mathrm{BO}(d-1)) \to \mathrm{Lift}(f, \partial f \text{ to } \mathrm{BO}(d), \mathrm{BO}(d-1) \text{ near } A)).$$

Then there is a weak equivalence

$$\mathrm{Sm}(M \text{ rel } A; \Sigma_0) \to \mathrm{Lift}(f, \partial f \text{ to } \mathrm{BO}(d), \mathrm{BO}(d-1) \text{ rel } f_0, \partial f_0) \tag{3.16}$$

if $\dim M \neq 4 \neq \dim \partial M$; where $f, \partial f$ classify the tangent microbundles of $M, \partial M$ and $f_0, \partial f_0$ are prescribed by the smooth structure Σ_0. For a proof see [KS77] pages 240-244.

3.3 Homotopy groups of Top/O

Having these very powerful classification theorems at our disposal we can compute homotopy groups of Top/O. We first prove the following stabilization theorem.

Theorem 3.17. *For* $0 \le k \le m$ *and* $m \ge 6$ *we have* $\pi_{k+1}(\mathrm{Top}(m)/\mathrm{O}(m), \mathrm{Top}(m-1)/\mathrm{O}(m-1)) = 0$ *and hence the maps* $\mathrm{Top}(m)/\mathrm{O}(m) \to \mathrm{Top}/\mathrm{O}$ *are* $(m+2)$*-connected.*

Proof. We apply the relative classification theorem 3.16 to the manifold $I \times D^k \times \mathbb{R}^n$ where $1 + k + n = m$ with $A = I \times \partial D^k \times \mathbb{R}^n \cup 1 \times D^k \times \mathbb{R}^n$, Σ_0 given by the standard smooth structure on A, and $f_0, \partial f_0$ the constant maps. Then 3.16 tells us that

$$\pi_0 \,\mathrm{Sm}(I \times D^k \times \mathbb{R}^n \text{ rel } A; \Sigma_0) \simeq \pi_0 \,\mathrm{Lift}(f, \partial f \text{ to } \mathrm{BO}(m), \mathrm{BO}(m-1) \text{ rel } f_0, \partial f_0) \simeq$$

$$\simeq [(I \times D^k, \partial(I \times D^k), I \times \partial D^k \cup 1 \times D^k), (\mathrm{Top}(m)/\mathrm{O}(m), \mathrm{Top}(m-1)/\mathrm{O}(m-1), *)] \simeq$$

$$\simeq \pi_{k+1}(\mathrm{Top}(m)/\mathrm{O}(m), \mathrm{Top}(m-1)/\mathrm{O}(m-1)).$$

By the concordance implies isotopy theorem ([KS77] page 25), the left-hand side is zero, this completes the proof. □

Let Θ_k denote the group of smooth oriented homotopy k-spheres up to oriented diffeomorphism and the operation given by connected sum. We are finally ready the main theorem of this section which we will use later.

Theorem 3.18 ([KS77] 5.3 on page 247)**.** *The homotopy groups of* Top/O *are given by*

$$\pi_k \,\mathrm{Top}/\mathrm{O} \simeq 0 \text{ for } k = 0, 1, 2, 4,$$

$$\pi_3 \,\mathrm{Top}/\mathrm{O} \simeq \mathbb{Z}/2,$$

$$\pi_k \,\mathrm{Top}/\mathrm{O} \simeq \Theta_k \text{ for } k > 4.$$

Proof. For $k \le 4$, the proof is significantly more involved, relying on deep results from surgery theory concerning the classification of homotopy tori; therefore, we omit it. For $k \ge 5$ we have $\pi_k \,\mathrm{Top}(k)/\mathrm{O}(k) \simeq \pi_k \,\mathrm{Top}/\mathrm{O}$ by Theorem 3.17. Moreover, by the relative classification theorem 3.16, there is an isomorphism

$$\pi_k \,\mathrm{Top}(k)/\mathrm{O}(k) \simeq \pi_0 \,\mathrm{Sm}(S^k \text{ rel } D^k_-; \Sigma_0),$$

where D^k_- is the bottom hemisphere and Σ_0 is the standard smooth structure near it. There is a forgetful map

$$\alpha : \pi_0 \,\mathrm{Sm}(S^k \text{ rel } D^k_-; \Sigma_0) \to \Theta_k$$

$$S^k_\Sigma \mapsto S^k_\Sigma.$$

We claim that α is a bijection. For surjectivity, let X be an oriented homotopy sphere, by the h-cobordism theorem we can find an orientation preserving homeomorphism $h : X \to D^k \cup_\phi D^k$ where $\phi : S^{k-1} \to S^{k-1}$ is a diffeomorphism and the preimage of the bottom hemisphere under h is a smoothly embedded disc in X. Then the pushforward smooth structure on S^k under h

gives us the same element in Θ_k as X; moreover, this smooth structure is standard near the bottom hemisphere, thus α is surjective. For injectivity, let Σ, Σ' be two smooth structures on S^k which are standard near the bottom hemisphere, and assume that there is an orientation preserving diffeomorphism $f : S^k_\Sigma \to S^k_{\Sigma'}$. $f(D^k_-)$ is a smoothly embedded disc in $S^k_{\Sigma'}$; therefore, it is smoothly ambient isotopic to the bottom hemisphere. This isotopy gives us another diffeomorphism $f' : S^k_\Sigma \to S^k_{\Sigma'}$ which is equal to Id on the bottom hemisphere. We produce an isotopy $H : S^k \times I \to S^k$ from f' to Id by putting Alexander isotopy on the top hemisphere. H gives us a concordance between Σ and Σ' relative to the bottom hemisphere, so α is injective. □

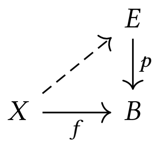
Remark 3.19. The groups of homotopy spheres Θ_k were studied by Kervaire and Milnor in [KM63]. A substantial amount of information is known about these groups. For example, they are finite, which implies that there exist only finitely many non-diffeomorphic smooth structures on compact topological manifolds of dimension at least 6.

Remark 3.20. Kirby and Siebenmann also developed the same theory for PL manifolds. One can define spaces $\mathrm{BPL}(m)$ which classify PL microbundles and by investigating PL structures on spheres one can prove that $\mathrm{Top}/\mathrm{PL} \simeq K(\mathbb{Z}/2, 3)$. Such a huge difference compared to the smooth case is caused by the fact that the Poincaré conjecture is true in the PL category in dimensions at least 5. We will use this homotopy equivalence later because it provides a very convenient way of defining the Kirby-Siebenmann invariant.

4 Obstruction theory

In this section, we introduce standard notions of obstruction theory, such as Postnikov towers and k-invariants, and prove some basic facts about them. For a more complete treatment of the topic, one can consult [GJ09] or [Tho66].
Obstruction theory answers the following commonly occurring question:

Question 4.1. *Let $p : E \to B$ be a Serre fibration, and let $f : X \to B$ a map, does f lift to E?*



Under suitable assumptions on the space B, obstruction theory gives a sequence of "obstructions" in $H^{i+1}(X, \pi_i F)$ where F is a fiber of p. Each of these obstructions is defined if the previous one vanishes and a map lifts if all of them vanish. We will come back to this question at the end of the section.
There are two ways to set up obstruction theory: one by filtering X with skeleta, and another by introducing Moore-Postnikov factorization of p; we take the second approach in this thesis and start with the notion of Postnikov towers.

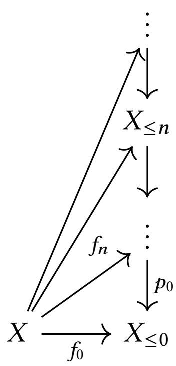
Definition 4.2. Let X be a connected space. A **Postnikov tower** of X is a sequence of spaces $X_{\leq n}$ equipped with maps

$$f_n : X \to X_{\leq n}$$

$$p_n : X_{\leq n+1} \to X_{\leq n}$$

for all $n \geq 0$, such that:

1. f_n induces an isomorphism on π_i for $i \leq n$.
2. $\pi_i X_{\leq n} = 0$ for $i > n$.
3. The diagram



commutes.

We also say that a Postnikov tower is **principal** if all p_n are principal $K(\pi_{n+1}X, n+1)$-fibrations.

Definition 4.3. Let X be a connected space with a principal Postnikov tower $\{X_{\leq n}\}$. Since all p_n are principal, $X_{\leq n}$ admit maps to $K(\pi_{n+1}X, n+2)$, which classify these fibrations. This is equivalent to a collection of cohomology classes $k_n \in H^{n+2}(X_{\leq n}, \pi_{n+1}X)$. We call these classes the **k-invariants** of X.

Postnikov towers exist in great generality.

Theorem 4.4 ([Hat02]). *Let X be a connected CW complex, then it admits a Postnikov tower $\{X_{\leq n}\}$.*

Proof. Set $X_{\leq 0} := *$ and define $f_0 : X \to X_{\leq 0}$ as the unique projection map $X \to *$. Now let $n \geq 1$, and let $\{\phi_i\}$ be a set of generators of $\pi_{n+1}X$. Attach $(n+2)$-cells to X along ϕ_i and denote the new space by X^{n+1}. By the cellular approximation theorem, the inclusion $X \to X^{n+1}$ induces an isomorphism on π_i for $i < n+1$ and $\pi_{n+1}X^{n+1} = 0$. Next, attach $(n+3)$-cells to X^{n+1} in a similar way, and denote the resulting space by X^{n+2}. Then $X \to X^{n+2}$ is an $(n+1)$-equivalence and $\pi_{n+1}X^{n+2} = \pi_{n+2}X^{n+2} = 0$. Continue by induction and set $X_{\leq n} := \bigcup_{i>n} X^i$, then the inclusion $f_n : X \to X_{\leq n}$ is $(n+1)$-connected and $\pi_i X_{\leq n} = 0$ for $i > n$.
We can find maps $p_n : X_{\leq n+1} \to X_{\leq n}$ making the diagrams

$$\begin{array}{ccc} & & X_{\leq n+1} \\ & \overset{f_{n+1}}{\nearrow} & \downarrow{\scriptstyle p_n} \\ X & \xrightarrow{f_n} & X_{\leq n} \end{array}$$

commutative for all $n \geq 0$ because $X_{\leq n+1}$ is obtained from X by attaching cells of dimension at least $n+3$ and $\pi_i X_{\leq n} = 0$ for $i > n$. Therefore, f_n extends to $X_{\leq n+1}$ and we denote this extension by p_n. □

Furthermore, Postnikov towers behave naturally with respect to maps between spaces.

Theorem 4.5 ([Hat02]). *Let $g : X \to Y$ be a map of connected CW complexes and let $\{X_{\leq n}\}$, $\{Y_{\leq n}\}$ be Postnikov towers of X, Y. Then there are maps $g_n : X_{\leq n} \to Y_{\leq n}$ such that the diagrams*

$$\begin{array}{ccc} X & \longrightarrow & X_{\leq n} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g_n} \\ Y & \longrightarrow & Y_{\leq n} \end{array} \qquad \begin{array}{ccc} X_{\leq n+1} & \xrightarrow{g_{n+1}} & Y_{\leq n+1} \\ \downarrow & & \downarrow \\ X_{\leq n} & \xrightarrow[g_n]{} & Y_{\leq n} \end{array}$$

commute up to homotopy.

Proof. Firstly, construct another Postnikov tower $\{X'_{\leq n}\}$ of X in the same way as in Theorem 4.4. There are maps $\alpha_n : X'_{\leq n} \to X_{\leq n}$ such that the diagrams

$$\begin{array}{ccc} & & X'_{\leq n} \\ & \nearrow & {\scriptstyle \alpha_n}\downarrow \\ X & \longrightarrow & X_{\leq n} \end{array}$$

commute, because $X'_{\leq n}$ is obtained from X by attaching cells of dimension at least $n+2$ and $\pi_i X_{\leq n} = 0$ for $i > n$. Moreover, maps α_n make the diagrams

$$\begin{array}{ccc} X'_{\leq n+1} & \xrightarrow{\alpha_{n+1}} & X_{\leq n+1} \\ \downarrow & & \downarrow \\ X'_{\leq n} & \xrightarrow{\alpha_n} & X_{\leq n+1} \end{array}$$

commute up to homotopy relative to X. A homotopy exists because $X'_{\leq n+1} \times I$ is obtained from $X'_{\leq n+1} \times 0 \cup X'_{\leq n+1} \times 1 \cup X \times I$ by attaching cells of dimension at least $n+3$. Therefore, we have a map of towers $\{X'_{\leq n}\} \xrightarrow{\alpha} \{X_{\leq n}\}$ under X. Similarly one constructs a map of towers $\{X'_{\leq n}\} \xrightarrow{\beta} \{Y_{\leq n}\}$ under X. The map of towers promised in the theorem statement is obtained by composing a homotopy inverse of α with β. □

Remark 4.6. Note that in the theorem above, we also proved that a Postnikov tower of a connected CW complex X is unique up to homotopy equivalence of towers.

As we have seen, every CW complex admits a Postnikov tower; however, not every tower can be refined to a principal one. For this to be true we need an additional assumption on X.

Theorem 4.7 ([Hat02]). *Let X be a connected CW complex such that $\pi_1 X$ is abelian and the action of $\pi_1 X$ on $\pi_n X$ is trivial for all $n > 0$ (such spaces are also called simple). Then X admits a Postnikov tower of principal fibrations $\{X_{\leq n}\}$.*

Proof. Start with any Postnikov tower $\{X_{\leq n}\}$ with all $p_n : X_{\leq n+1} \to X_{\leq n}$ being fibrations, any Postnikov tower can be refined to such a tower by replacing all p_n with fibrations; fibers of p_n have homotopy type of $K(\pi_n X, n)$ by the long exact sequence of homotopy groups. Notice that the action of $\pi_1 X_{\leq n}$ on $\pi_i X_{\leq n}$ can be identified the with action of $\pi_1 X$ on $\pi_i X$; therefore, it is trivial. Consider the cofiber $\operatorname{Cone}(p_n)$, we claim that it is $(n+1)$-connected and $\pi_{n+2} \operatorname{Cone}(p_n) \simeq \pi_{n+1} X$. Since $X_{\leq n+1} \to X_{\leq n}$ is $(n+1)$-connected, we may assume that $\operatorname{Cyl}(p_n)$ is obtained from $X_{\leq n+1}$ by attaching cells of dimension at least $n+2$, therefore $\pi_i \operatorname{Cone}(p_n) = 0$ for $i < n+2$. Consider the following commutative diagram

$$\begin{array}{ccc} \pi_{n+2}(\operatorname{Cyl}(p_n), X_{\leq n+1}) & \longrightarrow & \pi_{n+2} \operatorname{Cone}(p_n) \\ h\downarrow & & h\downarrow \\ H_{n+2}(\operatorname{Cyl}(p_n), X_{\leq n+1}) & \longrightarrow & H_{n+2} \operatorname{Cone}(p_n) \end{array}$$

with vertical arrows given by the Hurewicz homomorphism and horizontal by the quotient maps. The right vertical map is an isomorphism by the Hurewicz theorem. The bottom horizontal map is an isomorphism by excision. The left vertical arrow is an isomorphism by the relative Hurewicz theorem because $H_i(\operatorname{Cyl}(p_n), X_{\leq n+1}) = 0$ for $i < n+2$ and the action of $\pi_1 X_{\leq n+1}$ is trivial on $\pi_i(\operatorname{Cyl}(p_n), X_{\leq n+1})$. Therefore, we have an isomorphism $\pi_{n+1} X \simeq \pi_{n+2}(\operatorname{Cyl}(p_n), X_{\leq n+1}) \simeq \pi_{n+2} \operatorname{Cone}(p_n)$, we also conclude that $(\operatorname{Cone}(p_n))_{\leq n+2}$ has the homotopy type of $K(\pi_{n+1} X, n+2)$. We claim that the composition

$$X_{\leq n} \to \operatorname{Cone}(p_n) \to \operatorname{Cone}(p_n)_{\leq n+2}$$

represents n-th k-invariant of X. Write

$$F_{n+1} \coloneqq \mathrm{fib}(X_{\leq n} \to \mathrm{Cone}(p_n) \to \mathrm{Cone}(p_n)_{\leq n+2}),$$

then $X_{\leq n+1}$ admits a preferred map to F_{n+1} given by the preferred null-homotopy of the composition

$$X_{\leq n+1} \xrightarrow{p_n} X_{\leq n} \to \mathrm{Cone}(p_n).$$

Consider a map of long exact sequences

$$\begin{array}{ccccccc} \pi_k(\mathrm{Cyl}(p_n), X_{\leq n+1}) & \longrightarrow & \pi_k X_{\leq n+1} & \longrightarrow & \pi_k X_{\leq n} & \longrightarrow & \pi_{k-1}(\mathrm{Cyl}(p_n), X_{\leq n+1}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \pi_k(\mathrm{Cone}(p_n)_{\leq n+2}) & \longrightarrow & \pi_k F_{n+1} & \longrightarrow & \pi_k X_{\leq n} & \longrightarrow & \pi_{k-1}(\mathrm{Cone}(p_n)_{\leq n+2}). \end{array}$$

The five-lemma implies that the map $X_{\leq n+1} \to F_{n+1}$ is a weak equivalence. Therefore, we can replace $X_{\leq n+1}$ with F_{n+1} in the tower and now $F_{n+1} \to X_{\leq n}$ is a principal $K(\pi_{n+1}X, n+1)$-fibration because it arises as the following pullback

$$\begin{array}{ccc} F_{n+1} & \longrightarrow & P_* K(\pi_{n+1}X, n+2) \\ \downarrow & \lrcorner & \downarrow \\ X_{\leq n} & \longrightarrow & K(\pi_{n+1}X, n+2), \end{array}$$

where P_* denotes the path space. We can do this procedure inductively starting with p_0 to replace the whole tower with a tower of principal fibrations. □

Note that the construction of k-invariants from the previous theorem also implies naturality of k-invariants.

Lemma 4.8. *Let $g : X \to Y$ be a map between simple, connected CW complexes, let $\{X_{\leq n}\}$ and $\{Y_{\leq n}\}$ be principal Postnikov towers of X and Y with maps $g_n : X_{\leq n} \to Y_{\leq n}$ such that the diagram of towers commutes up to homotopy, and let*

$$k_{n,X} : X_{\leq n} \to K(\pi_{n+1}X, n+2),$$

$$k_{n,Y} : Y_{\leq n} \to K(\pi_{n+1}Y, n+2)$$

be the k-invariants of X and Y. Then the diagram

$$\begin{array}{ccc} X_{\leq n} & \xrightarrow{k_{n,X}} & K(\pi_{n+1}X, n+2) \\ {\scriptstyle g_n}\downarrow & & \downarrow{\scriptstyle g_*} \\ Y_{\leq n} & \xrightarrow{k_{n,Y}} & K(\pi_{n+1}Y, n+2) \end{array} \tag{4.9}$$

commutes up to homotopy. Moreover, one can replace $X_{\leq n}$, $Y_{\leq n}$ with $\mathrm{fib}(k_{n,X})$, $\mathrm{fib}(k_{n,Y})$; and $g_{n+1} : X_{\leq n+1} \to Y_{\leq n+1}$ with the map induced on the horizontal fibers in the Diagram 4.9 without affecting the homotopy-commutativity of the tower diagram.

Proof. We have a commutative diagram

$$\begin{array}{ccc} X_{\leq n+1} & \xrightarrow{g_{n+1}} & Y_{\leq n+1} \\ {\scriptstyle p_n}\big\downarrow & & {\scriptstyle p'_n}\big\downarrow \\ X_{\leq n} & \xrightarrow{g_n} & Y_{\leq n}. \end{array} \tag{4.10}$$

Therefore, by the functoriality of the cone and Postnikov tower, we have the following homotopy-commutative diagram

$$\begin{array}{ccccccc} X_{\leq n} & \longrightarrow & \mathrm{Cone}(p_n) & \longrightarrow & \mathrm{Cone}(p_n)_{\leq n+2} & \xrightarrow{\simeq} & K(\pi_{n+1}X, n+2) \\ \big\downarrow{\scriptstyle g_n} & & \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle g_*} \\ Y_{\leq n} & \longrightarrow & \mathrm{Cone}(p'_n) & \longrightarrow & \mathrm{Cone}(p'_n)_{\leq n+2} & \xrightarrow{\simeq} & K(\pi_{n+1}Y, n+2). \end{array}$$

Moreover, the compositions in the top and bottom rows correspond to the k-invariants $k_{n,X}$ and $k_{n,Y}$; thus, Diagram 4.9 commutes up to homotopy. In addition, we have a commutative diagram

$$\begin{array}{ccccccc} X_{\leq n+1} & \xrightarrow[r]{} & \mathrm{fib}(k_{n,X}) & \longrightarrow & X_{\leq n} & \longrightarrow & \mathrm{Cone}(p_n) \\ {\scriptstyle g_{n+1}}\big\downarrow & & \big\downarrow & & {\scriptstyle g_n}\big\downarrow & & \big\downarrow \\ Y_{\leq n+1} & \xrightarrow[r']{} & \mathrm{fib}(k_{n,Y}) & \longrightarrow & Y_{\leq n} & \longrightarrow & \mathrm{Cone}(p'_n), \end{array}$$

where the maps r and r' are given by the preferred null-homotopies of the compositions

$$X_{\leq n+1} \xrightarrow{p_n} X_{\leq n} \to \mathrm{Cone}(p_n),$$

$$Y_{\leq n+1} \xrightarrow{p'_n} Y_{\leq n} \to \mathrm{Cone}(p'_n).$$

From Theorem 4.7, we also know that r and r' are homotopy equivalences; hence, we can replace $X_{\leq n+1}$, $Y_{\leq n+1}$ with $\mathrm{fib}(k_{n,X})$, $\mathrm{fib}(k_{n,Y})$ and g_{n+1} with the map induced by the functoriality of the fiber. □

We will make extensive use of this naturality property of k-invariants in later computations. Postnikov towers also have close counterparts – Whitehead towers; we will need them later.

Definition 4.11. Let X be a connected space. A **Whitehead tower** of X is a sequence of spaces $X_{\geq n}$ endowed with maps

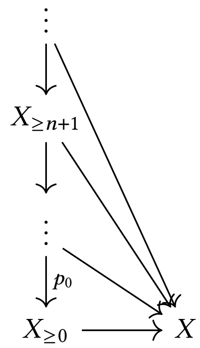
$$f_n : X_{\geq n} \to X,$$

$$p_n : X_{\geq n+1} \to X_{\geq n}$$

for all $n \geq 0$, such that:

1. f_n induces an isomorphism on π_i for $i \geq n$.
2. $\pi_i X_{\geq n} = 0$ for $i < n$.

3. The following diagram



commutes.

One way to construct a Whitehead tower of a connected CW complex X is to start with a Postnikov tower of X and take the homotopy fiber at each stage.
In the next sections, we will need a tool to compute k-invariants of spaces. The following lemma will be our main instrument.

Lemma 4.12 ([MT08] Chapter 11)**.** *Let X be a connected, simple space with a portion of its Postnikov tower:*

$$X_{\geq n+1} \longrightarrow X \longrightarrow X_{\leq n} \xrightarrow{k} K(\pi_{n+1}X, n+2)$$

and let $\gamma_{n+1} \in H^{n+1}(X_{\geq n+1}, \pi_{n+1}X)$ be the fundamental class corresponding to $\mathrm{Id} \in \mathrm{Hom}(\pi_{n+1}X, \pi_{n+1}X)$ under the isomorphisms

$$H^{n+1}(X_{\geq n+1}, \pi_{n+1}X) \to \mathrm{Hom}(H_{n+1}X_{\geq n+1}, \pi_{n+1}X) \to \mathrm{Hom}(\pi_{n+1}X, \pi_{n+1}X).$$

Then the k-invariant k is given by $k = \tau\gamma_{n+1}$, where τ is the transgression in the Serre spectral sequence of the fiber sequence $X_{\geq n+1} \to X \to X_{\leq n}$ with coefficients in $\pi_{n+1}X$.

Proof. We have the following commutative diagram (where pullback means homotopy pullback) from the definition of a principal Postnikov tower

$$\begin{array}{ccccc} & & X_{\leq n+1} & \xrightarrow{\alpha} & P_*K(\pi_{n+1}X, n+2) \\ & \overset{f_{n+1}}{\nearrow} & \downarrow & \lrcorner & \downarrow \\ X & \xrightarrow[f_n]{} & X_{\leq n} & \xrightarrow[k]{} & K(\pi_{n+1}X, n+2). \end{array}$$

By taking homotopy fibers, we can extend it to the following diagram

$$\begin{array}{ccccc} & & K(\pi_{n+1}X, n+1) & \xrightarrow{\beta} & K(\pi_{n+1}X, n+1) \\ & \overset{\gamma}{\nearrow} & \downarrow & & \downarrow \\ X_{\geq n+1} & & X_{\leq n+1} & \xrightarrow{\alpha} & P_*K(\pi_{n+1}X, n+2) \\ \downarrow & \overset{f_{n+1}}{\nearrow} & \downarrow & & \downarrow \\ X & \xrightarrow[f_n]{} & X_{\leq n} & \xrightarrow[k]{} & K(\pi_{n+1}X, n+2). \end{array}$$

Then $\beta \circ \gamma$ induces the identity on π_{n+1}, this implies commutativity of the following diagram

$$\begin{array}{ccc} H^{n+1}(X_{\geq n+1}, \pi_{n+1}X) & \xleftarrow{(\beta\circ\gamma)^*} & H^{n+1}(K(\pi_{n+1}X, n+1), \pi_{n+1}X) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(\pi_{n+1}X, \pi_{n+1}X) & \xleftarrow{\mathrm{Id}} & \mathrm{Hom}(\pi_{n+1}X, \pi_{n+1}X) \end{array}$$

and this in turn means that $(\beta \circ \gamma)^* \iota_{n+1} = \gamma_{n+1}$, where $\iota_{n+1} \in H^{n+1}(K(\pi_{n+1}X, n+1), \pi_{n+1}X)$ is the fundamental class. The following map of fiber sequences

$$\begin{array}{ccccc} X_{\geq n+1} & \longrightarrow & X & \xrightarrow{f_n} & X_{\leq n} \\ \downarrow{\scriptstyle \beta\circ\gamma} & & \downarrow{\scriptstyle \alpha\circ f_{n+1}} & & \downarrow{\scriptstyle k} \\ K(\pi_{n+1}X, n+1) & \longrightarrow & P_*K(\pi_{n+1}X, n+2) & \longrightarrow & K(\pi_{n+1}X, n+2) \end{array}$$

induces a map between Serre spectral sequences with coefficients in $\pi_{n+1}X$ and by the naturality of transgression we get

$$\tau\gamma_{n+1} = \tau((\beta \circ \gamma)^* \iota_{n+1}) = k^*(\tau\iota_{n+1}) = k^*\iota_{n+2} = k,$$

where $\iota_{n+2} \in H^{n+2}(K(\pi_{n+1}X, n+2), \pi_{n+1}X)$ is the fundamental class. The third equality follows from a direct check that $\tau\iota_{n+1} = \iota_{n+2}$; therefore, we are done. □

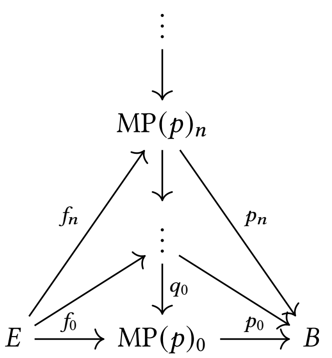
As we have already mentioned, there is a more general version of the Postnikov tower, which is called the Moore-Postnikov tower; we will need it in Section 7.

Definition 4.13. Let $p : E \to B$ be a map between connected spaces. A **Moore-Postnikov** tower of p is a sequence of spaces $\mathrm{MP}(p)_n$ endowed with maps

$$f_n : E \to \mathrm{MP}(p)_n,$$
$$p_n : \mathrm{MP}(p)_n \to B,$$
$$q_n : \mathrm{MP}(p)_{n+1} \to \mathrm{MP}(p)_n$$

for all $n \geq 0$, such that:

1. f_n induces isomorphisms on π_i for $i < n+1$ and a surjection on π_{n+1}.
2. p_n induces isomorphisms on π_i for $i > n+1$ and an injection on π_{n+1}.
3. The diagram



commutes.

We also say that a Moore-Postnikov tower is **principal** if all q_n are principal $K(\pi_{n+1}\operatorname{fib}(p), n+1)$-fibrations.

Similar results about existence, naturality, and k-invariants are also true for Moore-Postnikov towers, we state some of them without proof and give appropriate references.

Theorem 4.14. *Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a Serre fibration with B and F simply connected. Then p admits a principal Moore-Postnikov tower.*

For a proof, see Chapter 3 [Tho66]. Also, note that the condition of p being a fibration is not really a restriction, as any map can be replaced by a fibration. Now we define k-invariants of Moore-Postnikov towers and state a similar lemma which will help us to compute them later.

Definition 4.15. Let $p : E \to B$ be a map between connected spaces with a principal Moore-Postnikov tower $\{\mathrm{MP}(p)_n\}$. Since q_n are principal, $\mathrm{MP}(p)_n$ admit classifying maps to $K(\pi_{n+1}\operatorname{fib}(p), n+2)$, which correspond to cohomology classes $k_n \in H^{n+2}(\mathrm{MP}(p)_n, \pi_{n+1}\operatorname{fib}(p))$, we call these classes the **k-invariants** of p.

Lemma 4.16. *Let $F \to E \xrightarrow{p} B$ be a Serre fibration with B and F simply connected. Let*

$$\operatorname{fib}(f_n) \longrightarrow E \xrightarrow{f_n} \mathrm{MP}(p)_n \xrightarrow{k} K(\pi_{n+1}\operatorname{fib}(p), n+2)$$

be a portion of a principal Moore-Postnikov tower of p, and let $\gamma_{n+1} \in H^{n+1}(\operatorname{fib}(f_n), \pi_{n+1}\operatorname{fib}(p))$ be the fundamental class corresponding to $\mathrm{Id} \in \mathrm{Hom}(\pi_{n+1}\operatorname{fib}(p), \pi_{n+1}\operatorname{fib}(p))$ under the isomorphisms

$$H^{n+1}(\operatorname{fib}(f_n), \pi_{n+1}\operatorname{fib}(p)) \to \mathrm{Hom}(H_{n+1}\operatorname{fib}(f_n), \pi_{n+1}\operatorname{fib}(p)) \to \mathrm{Hom}(\pi_{n+1}\operatorname{fib}(p), \pi_{n+1}\operatorname{fib}(p)).$$

Then the k-invariant k is given by $k = \tau\gamma_{n+1}$, where τ is the transgression in the Serre spectral sequence of the fiber sequence $\operatorname{fib}(f_n) \to E \to \mathrm{MP}(p)_n$ with coefficients in $\pi_{n+1}\operatorname{fib}(p)$.

For a proof see Chapter 3 [Tho66].
We conclude the section by returning to the Question 4.1. Another very important fact about Moore-Postnikov towers is that there is a homotopy equivalence

$$E \xrightarrow{\simeq} \operatorname{holim}_n \mathrm{MP}(p)_n,$$

which implies that to determine whether $f : X \to B$ lifts to E we can attempt to lift it through the Moore-Postnikov tower step by step. Once we have a lift

$$\begin{array}{ccc} & & \mathrm{MP}(p)_n \\ & \nearrow & \downarrow \\ X & \longrightarrow & B, \end{array}$$

the k-invariant

$$X \to \mathrm{MP}(p)_n \to K(\pi_{n+1}\operatorname{fib}(p), n+2)$$

defines a cohomology class in X and this class vanishes if and only if we can lift f further to $\mathrm{MP}(p)_{n+1}$. If all these obstructions vanish, we can also lift f to E.

5 Characteristic classes

In this section, we recall information on characteristic classes of vector bundles, microbundles, and spherical fibrations which will be useful for us later. We show that the Stiefel-Whitney classes and the Euler class are defined for spherical fibrations and microbundles. We also state Wu's formula which describes the action of the Steenrod algebra on the Stiefel-Whitney classes. In the end, we define the first Pontryagin class and the Kirby-Siebenmann class for microbundles.

Definition 5.1. A **characteristic class** for m-dimensional vector bundles (microbundles, spherical fibrations) is a natural transformation from $\mathrm{Vect}_m(-)(\mathrm{Mic}_m(-), \mathrm{Sph}_m(-))$ to $H^k(-, A)$ as functors from the homotopy category of CW complexes to the category of sets, where $k \in \mathbb{N}$ is a natural number and A is an abelian group.

By Corollary 2.10 and Yoneda's lemma, we have

$$\mathrm{Nat}(\mathrm{Vect}_m(-), H^k(-, A)) \simeq \mathrm{Nat}([-, \mathrm{BO}(m)], H^k(-, A)) \simeq H^k(\mathrm{BO}(m), A).$$

Thus, it is enough to compute the cohomology of $\mathrm{BO}(m)$ to understand all characteristic classes of m-dimensional vector bundles. A similar statement holds for microbundles and spherical fibrations with $\mathrm{BO}(m)$ replaced by $\mathrm{BTop}(m)$ and $\mathrm{BG}(m)$ respectively by 2.11,2.13 (the statement for microbundles holds only if we work with microbundles over a sufficiently nice space for which the Kister-Mazur theorem holds).

Now we define some standard characteristic classes. Let $\xi = (E \xrightarrow{p} B)$ be a spherical fibration. We define its **Thom space** as $\mathrm{Th}(\xi) \coloneqq \mathrm{Cone}(p)$; the Thom isomorphism theorem still holds for spherical fibrations.

Theorem 5.2. *Let $\xi = (E \xrightarrow{p} B)$ be a $(m-1)$-dimensional spherical fibration over a CW complex B, then there is a Thom class $u \in H^m(\mathrm{Cyl}(p), E; \mathbb{Z}/2)$ and the composition*

$$H^k(B, \mathbb{Z}/2) \xrightarrow{p^*} H^k(\mathrm{Cyl}(p), \mathbb{Z}/2) \xrightarrow{-\cup u} H^{m+k}(\mathrm{Cyl}(p), E; \mathbb{Z}/2) \to \tilde{H}^{m+k}(\mathrm{Th}(\xi), \mathbb{Z}/2),$$

which we denote by Φ, is an isomorphism for all $k \geq 0$.
If, in addition, ξ is orientable and $u_{\mathbb{Z}} \in H^m(\mathrm{Cyl}(p), E; \mathbb{Z})$ is an integral Thom class then

$$\Phi_{\mathbb{Z}} : H^k(B, \mathbb{Z}) \xrightarrow{p^*} H^k(\mathrm{Cyl}(p), \mathbb{Z}) \xrightarrow{-\cup u_{\mathbb{Z}}} H^{m+k}(\mathrm{Cyl}(p), E; \mathbb{Z}) \to \tilde{H}^{m+k}(\mathrm{Th}(\xi), \mathbb{Z})$$

is also an isomorphism for all $k \geq 0$.

For a proof see Chapter 6 of [LM24]. Using the Thom isomorphism we can define the Stiefel-Whitney classes. Let $E_m \to \mathrm{BG}(m)$ be the universal $(m-1)$-spherical fibration.

Definition 5.3. Let $u \in \tilde{H}^d(\mathrm{Th}(E_m), \mathbb{Z}/2)$ be the Thom class, define the Stiefel-Whitney classes as

$$w_k \coloneqq \Phi^{-1} \mathrm{Sq}^k u \in H^k(\mathrm{BG}(m), \mathbb{Z}/2).$$

Moreover, we define w_k for microbundles and vector bundles as pullbacks of $w_k \in H^k(\mathrm{BG}(m), \mathbb{Z}/2)$ to $H^k(\mathrm{BTop}(m), \mathbb{Z}/2)$ and $H^k(\mathrm{BO}(m), \mathbb{Z}/2)$ along the maps induced by the inclusions $\mathrm{O}(m) \to \mathrm{Top}(m) \to \mathrm{G}(m)$.

It is clear that $w_k = 0$ for $k > m$ because $\mathrm{Sq}^k u = 0$ in this case. It is also known that the Stiefel-Whitney classes exhaust all the possible $\mathbb{Z}/2$ characteristic classes of vector bundles in the following sense

Theorem 5.4 (see [MS74]). *There is an isomorphism of rings*

$$\mathbb{Z}/2[w_1, w_2, \dots, w_m] \to H^*(\mathrm{BO}(m), \mathbb{Z}/2),$$

which associates to w_k the corresponding Stiefel-Whitney classes of the universal m-dimensional vector bundle.

However, microbundles and spherical fibrations have more characteristic classes with $\mathbb{Z}/2$ coefficients than just Stiefel-Whitney classes, as we will see later.
Stiefel-Whitney classes behave well with respect to the Whitney sum of vector bundles. Let ξ and η be m-dimensional and n-dimensional vector bundles over B. Then we have an equality

$$w(\xi \oplus \eta) = w(\xi)w(\eta) \in H^{\Pi}(B, \mathbb{Z}/2),$$

where $w(\xi) = \sum_k w_k(\xi) \in H^{\Pi}(B, \mathbb{Z}/2)$ is the total Stiefel-Whitney class (the same formula is also true for fiberwise join of spherical fibrations). This formula follows from Cartan's formula for the Steenrod squares and the fact that the Thom class of the Whitney sum $u_{\xi\oplus\eta} \in \tilde{H}^{m+n}(\mathrm{Th}(\xi \oplus \eta), \mathbb{Z}/2)$ corresponds to the product of the Thom classes of ξ and η via the isomorphisms

$$\tilde{H}^{m+n}(\mathrm{Th}(\xi \oplus \eta), \mathbb{Z}/2) \simeq \tilde{H}^{m+n}(\mathrm{Th}(\xi) \wedge \mathrm{Th}(\eta), \mathbb{Z}/2) \simeq \tilde{H}^{m}(\mathrm{Th}(\xi), \mathbb{Z}/2) \otimes \tilde{H}^{n}(\mathrm{Th}(\eta), \mathbb{Z}/2).$$

It is also possible to fully determine the action of the Steenrod algebra on the Stiefel-Whitney classes in $H^*(\mathrm{BG}(m), \mathbb{Z}/2)$. We introduce the following convention

$$\binom{a}{b} = \begin{cases} \frac{a!}{b!(a-b)!} & \text{if } a \geq b \geq 0, \\ 1 & \text{if } a = -1 \text{ and } b = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let ξ be a spherical fibration over a connected CW complex B, then we have:

Theorem 5.5. *(Wu's formula [Hsi63])*

$$\mathrm{Sq}^j w_k(\xi) = \sum_{t=0}^{j} \binom{k-j+t-1}{t} w_{j-t}(\xi) w_{k+t}(\xi)$$

for all $j, k \geq 0$.

We will use this formula in sections 6 and 7, so we write out more explicitly a couple of important cases for us:

$$\mathrm{Sq}^1 w_2 = w_1 w_2 + w_3,$$
$$\mathrm{Sq}^2 w_4 = w_2 w_4 + w_6.$$

Remark 5.6. A spherical fibration ξ is oriented if and only if $w_1(\xi)$ vanishes. One direction is straightforward: if ξ is oriented, then the mod 2 Thom class is the reduction of the integral Thom class, $u_{\mathbb{Z}/2} = \rho_2 u_{\mathbb{Z}}$, and therefore

$$w_1 = \Phi^{-1}(\mathrm{Sq}^1 u_{\mathbb{Z}/2}) = \Phi^{-1}(\mathrm{Sq}^1 \rho_2 u_{\mathbb{Z}}) = \Phi^{-1}(0) = 0.$$

The converse direction is harder, one can argue that the vanishing of w_1 is equivalent to ξ being trivializable over the 1-skeleton of B using obstruction theory and then proceed by induction over skeleta to show the existence of the integral Thom class (similar to the existence of the mod 2 Thom class).

By the remark above, we know that the universal spherical fibration $\tilde{E}_m \to \mathrm{BSG}(m)$ is orientable. Using this, we can define the Euler class $e \in H^m(\mathrm{BSG}(m), \mathbb{Z})$.

Definition 5.7. Let $u \in \tilde{H}^m(\mathrm{Th}(\tilde{E}_m), \mathbb{Z})$ be the integral Thom class of $\tilde{E}_m$, define the Euler class as

$$e := \Phi^{-1} u^2 \in H^m(\mathrm{BSG}(m), \mathbb{Z}).$$

Moreover, we define e for microbundles and vector bundles as pullbacks of $e \in H^m(\mathrm{BSG}(m), \mathbb{Z})$ to $H^m(\mathrm{BSTop}(m), \mathbb{Z})$ and $H^m(\mathrm{BSO}(m), \mathbb{Z})$ along the maps induced by the inclusions $\mathrm{SO}(m) \to \mathrm{STop}(m) \to \mathrm{SG}(m)$.

The Euler class also behaves well with respect to Whitney sums

$$e(\xi \oplus \eta) = e(\xi)e(\eta).$$

Moreover, $\rho_2 e(\xi) = w_m(\xi)$ for an oriented $(m-1)$-dimensional spherical fibration because

$$\Phi_{\mathbb{Z}/2}\rho_2 e = \Phi_{\mathbb{Z}/2}\rho_2\Phi_{\mathbb{Z}}^{-1}u_{\mathbb{Z}}^2 = u_{\mathbb{Z}/2}^2 = \mathrm{Sq}^m u_{\mathbb{Z}/2}.$$

With a bit more work one can also define the Pontryagin classes of vector bundles $p_k \in H^{4k}(\mathrm{BO}(m), \mathbb{Z})$ (see [MS74]), which satisfy the following properties:

1. $p_k(\xi) = 0$ if $\dim \xi < 2k$,
2. $p_k(\xi) = e(\xi)^2$ if $\dim \xi = 2k$ and ξ is oriented,
3. $\rho_2 p_k = w_{2k}^2$ for all $k \geq 0$,
4. $2p(\xi \oplus \eta) = 2p(\xi)p(\eta)$,
5. $p(\xi \oplus \underline{\mathbb{R}}) = p(\xi)$,

where $p(\xi) = \sum_k p_k(\xi) \in H^{\Pi}(B, \mathbb{Z})$ is the total Pontryagin class. The definition of the Pontryagin classes crucially uses vector bundle structure and they cannot be defined for microbundles and spherical fibrations in general.
The Stiefel-Whitney classes, the Euler class, and the Pontryagin classes give a full description of the integral cohomology of $\mathrm{BO}(d)$, which was computed by Brown in [Bro82]. We state a couple of cases which we will need later because the general result is rather convoluted:

$$H^*(\mathrm{BSO}(3), \mathbb{Z}) \simeq \mathbb{Z}[\beta_2 w_2, p_1]/(2\beta_2 w_2),$$

$$H^*(\mathrm{BSO}(4), \mathbb{Z}) \simeq \mathbb{Z}[\beta_2 w_2, p_1, e]/(2\beta_2 w_2),$$

where $\beta_2 : H^*(-, \mathbb{Z}/2) \to H^{*+1}(-, \mathbb{Z})$ is the Bockstein homomorphism.
Now we define a couple of classes which are specific to microbundles. Using the result of Kirby and Siebenmann that $\mathrm{Top}/\mathrm{PL} \simeq K(\mathbb{Z}/2, 3)$ (3.20) we can define a class named after them.

Definition 5.8. Define the Kirby-Siebenmann class $\mathrm{ks} \in H^4(\mathrm{BSTop}(m), \mathbb{Z}/2)$ as the composition

$$\mathrm{BSTop}(m) \to \mathrm{BSTop} \to \mathrm{B\,Top}/\mathrm{PL} \simeq K(\mathbb{Z}/2, 4).$$

The Kirby-Siebenmann class of a tangent microbundle of a topological manifold is an obstruction to the existence of a smooth structure. There are many manifolds M such that $\mathrm{ks}(\mathfrak{t}M) \neq 0$; for example, the famous $E8$ manifold.
As we have said, the definition of the Pontryagin classes uses vector bundle structure; however, the following theorem of Milgram allows us to define p_1 for microbundles.

Theorem 5.9 ([Mil88]). *There is a homotopy equivalence of 7-types* $\mathrm{BSTop}_{\le 7} \xrightarrow{\theta} \mathrm{BSO}_{\le 7} \times K(\mathbb{Z}/2, 4)$ *such that the composition*

$$\mathrm{BSO} \to \mathrm{BSTop} \to \mathrm{BSTop}_{\le 7} \xrightarrow{\theta} \mathrm{BSO}_{\le 7} \times K(\mathbb{Z}/2, 4) \xrightarrow{\mathrm{pr}_1} \mathrm{BSO}_{\le 7}$$

is 8-connected, and

$$\mathrm{BSTop} \to \mathrm{BSTop}_{\le 7} \xrightarrow{\theta} \mathrm{BSO}_{\le 7} \times K(\mathbb{Z}/2, 4) \xrightarrow{\mathrm{pr}_2} K(\mathbb{Z}/2, 4)$$

corresponds to the Kirby-Siebenmann class.

Definition 5.10. The theorem of Milgram implies that the map $\mathrm{BSO} \to \mathrm{BSTop}$ induces an isomorphism

$$H^4(\mathrm{BSTop}, \mathbb{Z}) \xrightarrow{\simeq} H^4(\mathrm{BSO}, \mathbb{Z}) \simeq \mathbb{Z}\langle p_1 \rangle.$$

Define $\tilde{p}_1 \in H^4(\mathrm{BSTop}, \mathbb{Z})$ as the preimage of $p_1 \in H^4(\mathrm{BSO}, \mathbb{Z})$ under this isomorphism. Define $\tilde{p}_1 \in H^4(\mathrm{BSTop}(m), \mathbb{Z})$ for $m > 0$ as the pullback of $\tilde{p}_1 \in H^4(\mathrm{BSTop}, \mathbb{Z})$ under the map $\mathrm{BSTop}(m) \to \mathrm{BSTop}$.

6 Postnikov tower of BSTop(4)

This section provides computations of the Postnikov towers of the spaces BSTop(4), BSO(4), BSG(4) through dimension 5. The main result of this section is

Theorem 6.1. *There is a homotopy equivalence of 5-types* $\mathrm{BSTop}(4)_{\leq 5} \xrightarrow{\theta} \mathrm{BSO}(4)_{\leq 5} \times K(\mathbb{Z}/2, 4)$ *such that the composition*

$$\mathrm{BSO}(4) \to \mathrm{BSTop}(4) \to \mathrm{BSTop}(4)_{\leq 5} \xrightarrow{\theta} \mathrm{BSO}(4)_{\leq 5} \times K(\mathbb{Z}/2, 4) \xrightarrow{\mathrm{pr}_1} \mathrm{BSO}(4)_{\leq 5}$$

is 6-connected, and

$$\mathrm{BSTop}(4) \to \mathrm{BSTop}(4)_{\leq 5} \xrightarrow{\theta} \mathrm{BSO}(4)_{\leq 5} \times K(\mathbb{Z}/2, 4) \xrightarrow{\mathrm{pr}_2} K(\mathbb{Z}/2, 4)$$

corresponds to the Kirby-Siebenmann class.

The strategy of the proof is as follows: first, we compute homotopy groups of these spaces and maps between them. Afterward, we patiently apply Lemma 4.12 to deduce information about the k-invariants of BSO(4) and BSG(4), and then compare them with those of BSTop(4) to prove Theorem 6.1.

6.1 Homotopy groups

We begin with a computation of homotopy groups of the aforementioned spaces in low dimensions.
The following lemma is a well-known folklore fact.

Lemma 6.2. SO(4) *is homeomorphic to* $\mathbb{R}P^3 \times S^3$.

Proof. Identify $\mathbb{R}P^3$ with the quotient space of a 3-dimensional ball D^3 of radius π with boundary points identified via the antipodal map. We describe a map

$$D^3 \to \mathrm{SO}(3)$$

which induces a homeomorphism $\mathbb{R}P^3 \cong \mathrm{SO}(3)$. We associate to a vector $v \in D^3$ a rotation in the counterclockwise direction by the angle $|v|$ in the plane orthogonal to v (the counterclockwise direction is well-defined because there is a preferred orientation in the orthogonal complement of v induced by v and the standard orientation of $\mathbb{R}^3$). It can be seen that this map is continuous by writing out the formulas explicitly. Moreover, it is surjective by Euler's rotation theorem and injectivity fails only at the antipodal points on the boundary of D^3. Therefore, this map induces a homeomorphism $\mathbb{R}P^3 \cong \mathrm{SO}(3)$.
Recall that there is a principal SO(3) bundle $\mathrm{SO}(4) \to S^3$, we claim that it admits a section. Indeed, view S^3 as a topological group using unit quaternions, left multiplication by a unit quaternion defines an element of SO(4). This gives us a section

$$s : S^3 \to \mathrm{SO}(4)$$

$$x \mapsto (t \mapsto xt).$$

Therefore, the principal bundle $\mathrm{SO}(4) \to S^3$ is isomorphic to the trivial one, this gives us the desired homeomorphism $\mathrm{SO}(4) \cong \mathrm{SO}(3) \times S^3 \cong \mathbb{R}P^3 \times S^3$. □

Now we proceed to the space $\mathrm{G}(4)$. Recall that there are fibrations $\mathrm{F}(m) \to \mathrm{G}(m+1) \to S^m$ (Lemma 3.1 in [MM79]), then we have

Lemma 6.3. *The fibration* $\mathrm{F}(3) \xrightarrow{\iota} \mathrm{G}(4) \xrightarrow{\mathrm{ev}} S^3$ *is homeomorphic to the trivial fibration.*

Proof. We view S^3 as unit quaternions. This fibration also admits a section given by left multiplication with a quaternion, but it does not imply triviality because the map $\mathrm{G}(4) \xrightarrow{\mathrm{ev}} S^3$ is not necessarily a fiber bundle. Therefore, we just construct two inverse homeomorphisms:

$$S^3 \times \mathrm{F}(3) \to \mathrm{G}(4)$$

$$(x, f) \mapsto (t \mapsto xf(t));$$

$$\mathrm{G}(4) \to S^3 \times \mathrm{F}(3)$$

$$f \mapsto (f(1), (t \mapsto (f(1))^{-1} f(t))).$$

□

Combining 6.2, 6.3, homotopy equivalence $\mathrm{F}(3) \simeq (\Omega^3 S^3)_{\pm \mathrm{Id}}$ (path components of the Id and a reflection), and the work of Toda [Tod63] we can compute homotopy groups of $\mathrm{BO}(4)$ and $\mathrm{BG}(4)$. We summarize these computations in the following table:

i	0	1	2	3	4
$\pi_i(\mathrm{O}(4))$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$
$\pi_i(\mathrm{G}(4))$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/12 \oplus \mathbb{Z}$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$

(6.4)

Remark 6.5. The identifications of the generators of the groups in the table above come from the splittings 6.2, 6.3. For example

$$\pi_3 \mathrm{O}(4) \simeq \pi_3(\mathrm{SO}(3) \times S^3) \simeq \pi_3 \mathbb{R}P^3 \times \pi_3 S^3 \simeq \mathbb{Z}\langle p \rangle \oplus \mathbb{Z}\langle \mathrm{Id}_{S^3} \rangle,$$

where $p : S^3 \to \mathbb{R}P^3$ is the quotient map.

We will need information about the maps induced on homotopy groups by the inclusion $\mathrm{SO}(4) \to \mathrm{SG}(4)$. Note that the maps $\mathrm{G}(m) \to \mathrm{G}(m+1)$ factor through $\mathrm{F}(m)$ since they are given by the suspension. We will also need the following results of Haefliger and Toda.

Theorem 6.6 ([Hae66] Remark 7.7)**.** *The map* $\mathrm{G}(m) \to \mathrm{F}(m)$ *is* $(2m-3)$*-connected.*

Theorem 6.7 ([Tod63] Propositions 5.6 and 5.8)**.**

- $\pi_6 S^3$ *is isomorphic to* $\mathbb{Z}/12$ *and we denote the generator of the 4-torsion part by* ν'.

- $\pi_7 S^3$ *is isomorphic to* $\mathbb{Z}/2$ *and its generator is given by* $\nu' \circ \Sigma^4\eta$, *where* η *is the Hopf map* $S^3 \to S^2$.

Using these facts we can prove:

Lemma 6.8. *The maps* $\pi_k(\mathrm{SO}(3)) \to \pi_k(\mathrm{SF}(3))$, *induced by the composition* $\mathrm{SO}(3) \to \mathrm{SG}(3) \to \mathrm{SF}(3)$ *are isomorphisms for* $k = 0, 1, 4$ *and are surjective for* $k = 3$.

Proof. The case $k = 0$ is trivial because all spaces are connected. For $k = 1$, consider the following commutative diagram of fiber sequences

$$\begin{array}{ccccc} \mathrm{SO}(2) & \longrightarrow & \mathrm{SO}(3) & \xrightarrow{\mathrm{ev}} & S^2 \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{SF}(2) & \longrightarrow & \mathrm{SG}(3) & \xrightarrow{\mathrm{ev}} & S^2. \end{array}$$

Recall that $\mathrm{SF}(2) \simeq (\Omega^2 S^2)_{\mathrm{Id}}$; thus, on the π_1 level of the long exact sequences of homotopy groups we have

$$\begin{array}{ccccccc} \mathbb{Z} & \xrightarrow[\cdot 2]{} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0 \\ \downarrow{\scriptstyle \simeq} & & \downarrow & & \downarrow & & \\ \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_1\,\mathrm{SG}(3) & \longrightarrow & 0. \end{array}$$

Moreover, the connecting homomorphism

$$\pi_2 S^2 \to \pi_1\,\mathrm{SF}(2) \simeq \pi_3 S^2$$

$$\mathbb{Z} \to \mathbb{Z}$$

is given by the Whitehead product $[\mathrm{Id}_{S^2}, -]$ by Theorem 3.2 in [Whi46], and hence it is also multiplication by 2. Therefore, $\pi_1\,\mathrm{SO}(3) \to \pi_1\,\mathrm{SF}(3)$ is an isomorphism by the discussion above combined with Theorem 6.6. For $k = 3$, the π_3-portion of the long exact sequences of homotopy groups has the following form

$$\begin{array}{ccccccc} \mathbb{Z}/2 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\simeq} & \mathbb{Z} \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \simeq} \\ \mathbb{Z}/2 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & \pi_3\,\mathrm{SG}(3) & \twoheadrightarrow & \mathbb{Z}, \end{array} \tag{6.9}$$

where the map $\mathbb{Z}/2 \to \mathbb{Z}/2$ in the bottom row is

$$\pi_4 S^2 \xrightarrow{[\mathrm{Id}_{S^2}, -]} \pi_5 S^2$$

by Theorem 3.2 in [Whi46]. This Whitehead product is trivial because it lies in the kernel of the suspension homomorphism

$$\Sigma : \pi_5 S^2 \to \pi_6 S^3$$

and this suspension homomorphism is injective. Indeed, $\pi_5 S^2$ is generated by $\eta \circ \Sigma\eta \circ \Sigma^2\eta$, which is stably nontrivial by Proposition 5 in Chapter 17 of [MT08]. Therefore, $\pi_3\,\mathrm{SG}(3) \simeq \mathbb{Z}/2 \times \mathbb{Z}$ and the composition $\pi_3\,\mathrm{SO}(3) \to \pi_3\,\mathrm{SG}(3) \to \pi_3\,\mathrm{SF}(3)$ identifies with $\mathbb{Z} \xrightarrow{\alpha} \mathbb{Z}/2 \times \mathbb{Z} \xrightarrow{\beta} \mathbb{Z}/12$, since $\pi_3\,\mathrm{SF}(3) \simeq \pi_3((\Omega^3 S^3)_{\mathrm{Id}}) \simeq \mathbb{Z}/12$. Therefore, we have to prove that $\beta \circ \alpha$ is surjective. β is surjective by Theorem 6.6; thus, $\beta((0,1)) = 1$ (up to an automorphism of $\mathbb{Z}/12$). Moreover, by the commutativity of (6.9), $\alpha(1) = (x, 1)$ for some $x \in \mathbb{Z}/2$. Now we consider all possible cases:

1. $\alpha(1) = (0,1), \beta((1,0)) = 0,$
2. $\alpha(1) = (1,1), \beta((1,0)) = 0,$
3. $\alpha(1) = (0,1), \beta((1,0)) = 6,$
4. $\alpha(1) = (1,1), \beta((1,0)) = 6.$

In cases (1)-(3) $(\beta \circ \alpha)(1) = 1$ and in the case (4) $(\beta \circ \alpha)(1) = 7$ and therefore the map is surjective.
Now we turn to the case $k = 4$. Consider the following commutative diagram, where the vertical maps are given by precomposition with the suspensions of the Hopf fibration (note that they are homomorphisms, however, this is not important for the argument)

$$\begin{array}{ccccccc}
\pi_3\,\mathrm{SO}(3) & \longrightarrow & \pi_3\,\mathrm{SF}(3) & \xrightarrow{\simeq} & \pi_3((\Omega^3 S^3)_{\mathrm{Id}}) & \xrightarrow{\simeq} & \pi_6 S^3 \\
\downarrow{\scriptstyle -\circ\Sigma\eta} & & \downarrow{\scriptstyle -\circ\Sigma\eta} & & \downarrow{\scriptstyle -\circ\Sigma\eta} & & \downarrow{\scriptstyle -\circ\Sigma^4\eta} \\
\pi_4\,\mathrm{SO}(3) & \longrightarrow & \pi_4\,\mathrm{SF}(3) & \xrightarrow{\simeq} & \pi_4((\Omega^3 S^3)_{\mathrm{Id}}) & \xrightarrow{\simeq} & \pi_7 S^3.
\end{array}$$

By Theorem 6.7 $\pi_7 S^3 \simeq \mathbb{Z}/2$ and the rightmost vertical map is surjective. Therefore, by the surjectivity of $\pi_3\,\mathrm{SO}(3) \to \pi_3\,\mathrm{SF}(3)$ and the commutativity of the diagram, the map $\pi_4\,\mathrm{SO}(3) \to \pi_4\,\mathrm{SF}(3)$ is also surjective and hence bijective because both groups are $\mathbb{Z}/2$. □

Proposition 6.10. *The maps $\pi_k\,\mathrm{SO}(4) \to \pi_k\,\mathrm{SG}(4)$, induced by the inclusion are isomorphisms for $k = 0, 1, 4$ and are surjective for $k = 3$.*

Proof. Consider the following commutative diagram of fiber sequences

$$\begin{array}{ccccc}
\mathrm{SO}(3) & \longrightarrow & \mathrm{SO}(4) & \longrightarrow & S^3 \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{SF}(3) & \longrightarrow & \mathrm{SG}(4) & \longrightarrow & S^3.
\end{array}$$

The statement follows from the corresponding portions of the long exact sequence of homotopy groups, Lemma 6.8, and the splittings 6.2, 6.3. Moreover, for $k = 3$, the corresponding portion of the long exact sequences has the following form

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z}/12 & \longrightarrow & \mathbb{Z}/12 \oplus \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0.
\end{array} \tag{6.11}$$

Notice that the diagram of splittings given by left multiplication by a unit quaternion is commutative

$$\begin{array}{ccc} SO(4) & \xrightarrow{\qquad\qquad} & SG(4) \\ & {\scriptstyle s}\nwarrow \quad \nearrow{\scriptstyle s'} & \\ & S^3. & \end{array}$$

This implies that the middle vertical arrow in (6.11) is the direct sum of the vertical arrows on the sides. □

We now know the relevant homotopy groups of O(4) and G(4), as well as the maps between them induced by the inclusion. This will help us to compute homotopy groups of Top(4), since the map O(4) → G(4) factors through Top(4). To proceed with this computation, we need the following theorem of Quinn (compare with 3.17).

Theorem 6.12 ([FQ90] Theorem 8.7A)**.** *The stabilization map* Top(4)/O(4) → Top/*O is 5-connected.*

Using it we first deduce

Lemma 6.13. *The stabilization maps* G(4)/O(4) → G/O *and* G(4)/Top(4) → G/Top *are 5-connected.*

Proof. From Table 6.4 and Proposition 6.10, we deduce that $\pi_i\,G(4)/O(4)$ is isomorphic to $0, 0, \mathbb{Z}/2, 0, \mathbb{Z}$ for i from 0 to 4. The corresponding stable groups are the same by the computations of the stable J-homomorphism O → G (see, for example, [Ran02] Remark 9.22) and in addition $\pi_5\,G/O = 0$. The map O(4) → O is 3-connected and G(4) → G is also 3-connected because it is a composition of a 5-connected map G(4) → F(4) (Theorem 6.6) and a 3-connected map F(4) → G. Now compare the long exact sequences of homotopy groups of the fiber sequences

$$\begin{array}{ccccc} O(4) & \longrightarrow & G(4) & \longrightarrow & G(4)/O(4) \\ \downarrow & & \downarrow & & \downarrow \\ O & \xrightarrow{\ J\ } & G & \longrightarrow & G/O\,. \end{array}$$

From this we deduce that the maps

$$\pi_k\,G(4)/O(4) \to \pi_k\,G/O$$

are isomorphisms for $k = 0, 1, 2, 3$ and are surjective for $k = 5$. We are only left to consider the case $k = 4$, we have the following exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{1\mapsto(12,0)} & \mathbb{Z}\oplus\mathbb{Z} & \xrightarrow[(0,1)\mapsto(0,1)]{(1,0)\mapsto(1,0)} & \mathbb{Z}/12\oplus\mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle (1,0)\mapsto 2} & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}/24 & \longrightarrow & 0, \end{array}$$

where the middle vertical map

$$\pi_3\,\mathrm{O}(4) \to \pi_3\,\mathrm{O}$$

$$\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$$

is $(a, b) \mapsto 2a + b$ by (2.1), (2.2) in [Tam57]. Therefore, by commutativity, the left vertical arrow ($\pi_4\,\mathrm{G}(4)/\mathrm{O}(4) \to \pi_4\,\mathrm{G}/\mathrm{O}$) is an isomorphism and thus $\mathrm{G}(4)/\mathrm{O}(4) \to \mathrm{G}/\mathrm{O}$ is 5-connected. The statement about $\mathrm{G}(4)/\mathrm{Top}(4) \to \mathrm{G}/\mathrm{Top}$ follows from the long exact sequences of the fiber sequences

$$\begin{array}{ccccc} \mathrm{Top}(4)/\mathrm{O}(4) & \longrightarrow & \mathrm{G}(4)/\mathrm{O}(4) & \longrightarrow & \mathrm{G}(4)/\mathrm{Top}(4) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Top}/\mathrm{O} & \longrightarrow & \mathrm{G}/\mathrm{O} & \longrightarrow & \mathrm{G}/\mathrm{Top} \end{array}$$

by applying the 5-lemma together with the first part of this lemma and the theorem of Quinn. □

Remark 6.14. Note that the stable groups are known: $\pi_3\,\mathrm{G}/\mathrm{O} \simeq \pi_3\,\mathrm{G}/\mathrm{Top} \simeq 0$, $\pi_4\,\mathrm{G}/\mathrm{O} \simeq \mathbb{Z}$, $\pi_4\,\mathrm{G}/\mathrm{Top} \simeq \mathbb{Z}$. Moreover, the map

$$\mathbb{Z} \simeq \pi_4\,\mathrm{G}/\mathrm{O} \to \pi_4\,\mathrm{G}/\mathrm{Top} \simeq \mathbb{Z}$$

is the multiplication by 2 (see, for example, [KS77] page 318). Lemma 6.13 implies the same statements for the unstable spaces.

The following theorem should be known to experts. The author learned it from Manuel Krannich and Alexander Kupers; however, it does not seem like it has been written up anywhere.

Theorem 6.15. *Homotopy groups $\pi_k\,\mathrm{Top}(4)$ for $k < 5$ are given by the following table*

i	0	1	2	3	4
$\pi_i(\mathrm{Top}(4))$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$

Proof. The statement for $k = 0, 1, 2$ follows from Theorem 3.18 and Theorem 6.12. For $k = 4$, consider the fiber sequence

$$\mathrm{O}(4) \to \mathrm{Top}(4) \to \mathrm{Top}(4)/\mathrm{O}(4),$$

the π_4-portion of the induced long exact sequence of homotopy groups can be identified with

$$\mathbb{Z}/2 \oplus \mathbb{Z}/2 \to \pi_4(\mathrm{Top}(4)) \to 0.$$

Moreover, by Proposition 6.10, the composition $\pi_4\,\mathrm{O}(4) \to \pi_4\,\mathrm{Top}(4) \to \pi_4\,\mathrm{G}(4)$ is an isomorphism, therefore $\pi_4\,\mathrm{O}(4) \to \pi_4\,\mathrm{Top}(4)$ has to be an isomorphism as well. For $k = 3$, examining the long exact sequence leads to the following extension problem

$$0 \to \mathbb{Z} \oplus \mathbb{Z} \to \pi_3\,\mathrm{Top}(4) \to \mathbb{Z}/2 \to 0.$$

Kirby and Siebenmann show that an analogous sequence in the stable case splits ([KS77] page 318), we make a similar argument to show that it also splits in the unstable case. Assume that $\pi_3 \operatorname{Top}(4)$ is isomorphic to $\mathbb{Z}\oplus\mathbb{Z}$ to arrive at a contradiction. Consider the commutative diagram with horizontal rows being fiber sequences

$$\begin{array}{ccccc} \mathrm{O}(4) & \longrightarrow & \mathrm{G}(4) & \longrightarrow & \mathrm{G}(4)/\mathrm{O}(4) \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Top}(4) & \longrightarrow & \mathrm{G}(4) & \longrightarrow & \mathrm{G}(4)/\operatorname{Top}(4), \end{array}$$

by Table (6.4) and Remark 6.14 we get the following diagram of homotopy groups in dimensions 3 and 4 with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{1\mapsto(12,0)} & \mathbb{Z}\oplus\mathbb{Z} & \xrightarrow[(0,1)\mapsto(0,1)]{(1,0)\mapsto(1,0)} & \mathbb{Z}/12\oplus\mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \cdot 2} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \mathrm{Id}} & & \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\alpha} & \pi_3 \operatorname{Top}(4) & \xrightarrow{\beta} & \mathbb{Z}/12\oplus\mathbb{Z} & \longrightarrow & 0. \end{array} \tag{6.16}$$

We assumed that $\pi_3 \operatorname{Top}(4) \simeq \mathbb{Z}\oplus\mathbb{Z}$, then $\alpha(1) = (a, b)$, $\gamma((1,0)) = (a_1, b_1)$ for some integers a, b, a_1, b_1. By commutativity of the left square we get $(a, b) = (6a_1, 6b_1)$, but by commutativity of the right square and exactness of the bottom row $(0,0) = \beta((a,b)) = 6\beta(a_1, b_1) = (6,0)$. Therefore, we get a contradiction and $\pi_3 \operatorname{Top}(4)$ is isomorphic to $\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}/2$. □

Remark 6.17. Also, notice that the generator of 2-torsion in $\pi_3 \operatorname{Top}(4)$ has to be mapped to a nontrivial element in $\pi_3 \mathrm{G}(4)$ (in fact, it has to be $(6,0)$) by the exactness of the bottom row in (6.16). Indeed, if it was mapped to a trivial element, then it would be the only element in the image of the map $\alpha : \mathbb{Z} \to \pi_3 \operatorname{Top}(4) \simeq \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}/2$, which contradicts injectivity of α.

We now know homotopy groups of $\operatorname{Top}(4)$ and we are almost ready to start the computation of the k-invariants, but first, we need to introduce one more space which we call $\mathrm{BTSG}(4)$ following Milgram's notation in [Mil88] for the stable case.

Lemma 6.18. *The 3-type* $\mathrm{BSG}(4)_{\leq 3}$ *is homotopy equivalent to* $K(\mathbb{Z}/2, 2) \times K(\mathbb{Z}/2, 3)$.

Proof. We only need to prove that the first k-invariant k_1 of $\mathrm{BSG}(4)$ vanishes. Consider its Postnikov tower

$$\begin{array}{ccccccc} & & & \nearrow & \mathrm{BSG}(4)_{\leq 3} & & \\ & & & & \downarrow & & \\ \mathrm{BSG}(4)_{\geq 3} & \longrightarrow & \mathrm{BSG}(4) & \xrightarrow{w_2} & K(\mathbb{Z}/2, 2) & \xrightarrow{k_1} & K(\mathbb{Z}/2, 4). \end{array}$$

Where w_2 denotes the second Stiefel-Whitney class. To show that the first map in the Postnikov tower is given by w_2, it suffices to see that there are spherical fibrations with nontrivial w_2. By Lemma 4.12, k_1 is equal to the transgression of the fundamental class in the Serre spectral sequence of the fiber sequence

$$\mathrm{BSG}(4)_{\geq 3} \longrightarrow \mathrm{BSG}(4) \longrightarrow K(\mathbb{Z}/2, 2).$$

To compute this transgression, we can use the Serre exact sequence of the same spectral sequence:

$$H^3(\mathrm{BSG}(4),\mathbb{Z}/2) \longrightarrow H^3(\mathrm{BSG}(4)_{\geq 3},\mathbb{Z}/2) \xrightarrow{\tau} H^4(K(\mathbb{Z}/2,2),\mathbb{Z}/2) \longrightarrow H^4(\mathrm{BSG}(4),\mathbb{Z}/2).$$

There is an isomorphism

$$H^4(K(\mathbb{Z}/2,2),\mathbb{Z}/2) \simeq \mathbb{Z}/2\langle \gamma_2^2\rangle,$$

where γ_2 is the generator of $H^2(K(\mathbb{Z}/2,2),\mathbb{Z}/2)$; therefore, the map $H^4(K(\mathbb{Z}/2,2),\mathbb{Z}/2) \to H^4(\mathrm{BSG}(4),\mathbb{Z}/2)$ sends γ_2^2 to w_2^2. There are spherical fibrations with $w_2^2 \neq 0$, thus the map is nontrivial and consequently $\tau = 0$. Therefore $k_1 = 0$. □

By Lemma 6.18, $H^3(\mathrm{BSG}(4),\mathbb{Z}/2)$ is isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ with one of the generators given by w_3 because there are spherical fibrations with nontrivial w_3. Also, $H^3(\mathrm{BSTop}(4),\mathbb{Z}/2)$ is isomorphic to $\mathbb{Z}/2$ with the generator given by w_3 because the map $\mathrm{BSO}(4) \to \mathrm{BSTop}(4)$ is 3-connected and there are microbundles with nontrivial w_3. Denote the generator of the kernel of the map

$$H^3(\mathrm{BSG}(4),\mathbb{Z}/2) \to H^3(\mathrm{BSTop}(4),\mathbb{Z}/2)$$

by λ. Define $\mathrm{BTSG}(4)$ to be the homotopy fiber of the map

$$\lambda : \mathrm{BSG}(4) \to K(\mathbb{Z}/2,3).$$

Then the composition

$$\mathrm{BSTop}(4) \to \mathrm{BSG}(4) \xrightarrow{\lambda} K(\mathbb{Z}/2,3)$$

is null-homotopic by the definition of λ. This tells us that we have an induced map $\mathrm{BSTop}(4) \to \mathrm{BTSG}(4)$ (there are exactly two different maps up to homotopy since $H^2(\mathrm{BSTop}(4),\mathbb{Z}/2) = \mathbb{Z}/2$, the latter argument does not depend on which one we pick).
We claim that λ pairs nontrivially with the generator of $\pi_3\,\mathrm{BSG}(4)$. Indeed, this follows from the fact that λ has to have a component coming from the generator $\gamma_3 \in H^3(K(\mathbb{Z}/2,3),\mathbb{Z}/2)$ under the composition

$$\mathrm{BSG}(4) \to \mathrm{BSG}(4)_{\leq 3} \simeq K(\mathbb{Z}/2,2) \times K(\mathbb{Z}/2,3)$$

because otherwise λ would be equal to $\mathrm{Sq}\, w_2 = w_3 + w_1 w_2 = w_3$ by Wu's formula; and the latter maps nontrivially to $\mathrm{BSTop}(4)$. Then a computation with the long exact sequences of homotopy groups shows that $\mathrm{BTSG}(4)$ has the same homotopy groups as $\mathrm{BSG}(4)$ except for $\pi_3\,\mathrm{BTSG}(4) = 0$; and $\mathrm{BSTop}(4) \to \mathrm{BTSG}(4)$ induces the same maps on homotopy groups as $\mathrm{BSTop}(4) \to \mathrm{BSG}(4)$. We collect the computations we have done in the following table

i	0	1	2	3	4	5
$\pi_i(\mathrm{BSO}(3))$	0	0	$\mathbb{Z}/2$	0	$\mathbb{Z}$	$\mathbb{Z}/2$
$\pi_i(\mathrm{BSO}(4))$	0	0	$\mathbb{Z}/2$	0	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$
$\pi_i(\mathrm{BSTop}(4))$	0	0	$\mathbb{Z}/2$	0	$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$
$\pi_i(\mathrm{BSG}(4))$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/12 \oplus \mathbb{Z}$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$
$\pi_i(\mathrm{BTSG}(4))$	0	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/12 \oplus \mathbb{Z}$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$

(6.19)

Remark 6.20. We also already know all the maps in this table; mainly from Remark 6.5, Theorem 6.15, and Proposition 6.10. Namely, the maps on π_2 are the identity. The maps on π_4 are given by the following matrices

$$\begin{array}{ll}
\pi_4\,\mathrm{BSO}(3) \to \pi_4\,\mathrm{BSO}(4) & \begin{pmatrix}1\\0\end{pmatrix}\\
\pi_4\,\mathrm{BSO}(4) \to \pi_4\,\mathrm{BSTop}(4) & \begin{pmatrix}1&0\\0&1\\0&0\end{pmatrix}\\
\pi_4\,\mathrm{BSTop}(4) \to \pi_4\,\mathrm{BSG}(4) & \begin{pmatrix}1&0&6\\0&1&0\end{pmatrix}\\
\pi_4\,\mathrm{BSG}(4) \to \pi_4\,\mathrm{BTSG}(4) & \begin{pmatrix}1&0\\0&1\end{pmatrix}
\end{array}$$

The map $\pi_5\,\mathrm{BSO}(3) \to \pi_5\,\mathrm{BSO}(4)$ is the inclusion of the first summand and the other maps on π_5 are the identity.

6.2 k-invariants

After finishing the computation of all the relevant homotopy groups we can proceed to the computation of k-invariants. We begin with BSO(3) (a similar computation for this space is performed in [AW14]; however, our method is slightly different). From (6.19), we conclude that the first two stages of the Postnikov tower of BSO(3) have the following form:

$$\begin{array}{ccccc}
 & & \mathrm{BSO}(3)_{\leq 4} & \xrightarrow{k_2} & K(\mathbb{Z}/2,6)\\
 & \nearrow & \downarrow & & \\
\mathrm{BSO}(3) & \xrightarrow{w_2} & K(\mathbb{Z}/2,2) & \xrightarrow{k_1} & K(\mathbb{Z},5).
\end{array}$$

To compute the first k-invariant, we apply Lemma 4.12; thus, we need to understand the image of the transgression of the fundamental class in the spectral sequence for the fiber sequence

$$\mathrm{BSO}(3)_{\geq 4} \longrightarrow \mathrm{BSO}(3) \xrightarrow{w_2} K(\mathbb{Z}/2,2).$$

In order to do this, we look at the corresponding portion of the Serre exact sequence

$$H^4(\mathrm{BSO}(3),\mathbb{Z}) \longrightarrow H^4(\mathrm{BSO}(3)_{\geq 4},\mathbb{Z}) \xrightarrow{\tau} H^5(K(\mathbb{Z}/2,2),\mathbb{Z}) \longrightarrow H^5(\mathrm{BSO}(3),\mathbb{Z}).$$

There are isomorphisms (see Appendix A for more details):

$$H^5(K(\mathbb{Z}/2,2),\mathbb{Z}) \simeq \mathbb{Z}/4\langle \beta_4 P\gamma_2\rangle,$$

$$H^5(\mathrm{BSO}(3),\mathbb{Z}) = 0.$$

Therefore, τ maps a generator of $H^4(\mathrm{BSO}(3)_{\geq 4},\mathbb{Z}) \simeq \mathbb{Z}$ to a generator of $\mathbb{Z}/4\langle \beta_4 P\gamma_2\rangle$, so $k_1 = \pm\beta_4 P\gamma_2$. To proceed with the second k-invariant, we first need to compute $H^6(\mathrm{BSO}(3)_{\leq 4},\mathbb{Z}/2)$. For this, we consider the Serre spectral sequence of the fiber sequence

$$\mathrm{BSO}(3)_{\leq 4} \longrightarrow K(\mathbb{Z}/2,2) \xrightarrow{\pm\beta_4 P\gamma_2} K(\mathbb{Z},5)$$

with coefficients in $\mathbb{Z}/2$.

For convenience we also copy tables with cohomology groups of the relevant Eilenberg-Maclane spaces from the Appendix A.

$H^i(K(\mathbb{Z}, n), \mathbb{Z}/2)$	$i = 3$	4	5	6	7	8
$n = 5$	0	0	$\langle \rho_2 \iota_5 \rangle$	0	$\langle \mathrm{Sq}^2 \rho_2 \iota_5 \rangle$	$\langle \mathrm{Sq}^3 \rho_2 \iota_5 \rangle$
$H^i(K(\mathbb{Z}/2, n), \mathbb{Z}/2)$	$i = 3$	4	5	6	7	8
$n = 2$	$\langle \mathrm{Sq}^1 \gamma_2 \rangle$	$\langle \gamma_2^2 \rangle$	$\langle \gamma_2 \mathrm{Sq}^1 \gamma_2, \mathrm{Sq}^2 \mathrm{Sq}^1 \gamma_2 \rangle$	$\langle \gamma_2^3, (\mathrm{Sq}^1 \gamma_2)^2 \rangle$	$\langle \gamma_2 \mathrm{Sq}^2 \mathrm{Sq}^1 \gamma_2, \gamma_2^2 \mathrm{Sq}^1 \gamma_2 \rangle$	$\langle \gamma_2^4, \gamma_2 (\mathrm{Sq}^1 \gamma_2)^2, \mathrm{Sq}^1 \gamma_2 \mathrm{Sq}^2 \mathrm{Sq}^1 \gamma_2 \rangle$

$q \backslash p$	0	1	2	3	4	5	6	7
6	$\gamma_2'^3, (\mathrm{Sq}^1 \gamma_2')^2, t$	0						
5	$\gamma_2' \mathrm{Sq}^1 \gamma_2'$	0	0					
4	$\gamma_2'^2$	0	0	0				
3	$\mathrm{Sq}^1 \gamma_2'$	0	0	0	0			
2	γ_2'	0	0	0	0	$\gamma_2' \rho_2 \iota_5$		
1	0	0	0	0	0	0	0	
0	$\mathbb{Z}/2$	0	0	0	0	$\rho_2 \iota_5$	0	$\mathrm{Sq}^2 \rho_2 \iota_5$

(6.21)

$$H^p(K(\mathbb{Z}, 5), H^q(\mathrm{BSO}(3)_{\leq 4}, \mathbb{Z}/2)) \implies H^{p+q}(K(\mathbb{Z}/2, 2), \mathbb{Z}/2).$$

Since the coefficients are $\mathbb{Z}/2$, we only write generators of groups in the spectral sequence for brevity. From convergence, multiplicativity, and naturality we get

$$H^2(\mathrm{BSO}(3)_{\leq 4}, \mathbb{Z}/2) = \mathbb{Z}/2\langle \gamma_2' \rangle,$$
$$H^3(\mathrm{BSO}(3)_{\leq 4}, \mathbb{Z}/2) = \mathbb{Z}/2\langle \mathrm{Sq}^1 \gamma_2' \rangle,$$
$$H^4(\mathrm{BSO}(3)_{\leq 4}, \mathbb{Z}/2) = \mathbb{Z}/2\langle \gamma_2'^2 \rangle.$$

Where γ_2' is the image of $\gamma_2 \in H^2(K(\mathbb{Z}/2, 2), \mathbb{Z}/2)$. We claim that the edge homomorphisms

$$H^5(K(\mathbb{Z}, 5), \mathbb{Z}/2) \to H^5(K(\mathbb{Z}/2, 2), \mathbb{Z}/2), \tag{6.22}$$
$$H^7(K(\mathbb{Z}, 5), \mathbb{Z}/2) \to H^7(K(\mathbb{Z}/2, 2), \mathbb{Z}/2), \tag{6.23}$$
$$H^7(K(\mathbb{Z}/2, 2), \mathbb{Z}/2) \to H^7(\mathrm{BSO}(3)_{\leq 4}, \mathbb{Z}/2) \tag{6.24}$$

are nontrivial. For (6.22) we have to compute $\rho_2\beta_4 P\gamma_2$, consider the Bockstein exact sequence

$$H^5(K(\mathbb{Z}/2,2),\mathbb{Z}) \xrightarrow{\cdot 2} H^5(K(\mathbb{Z}/2,2),\mathbb{Z}) \xrightarrow{\rho_2} H^5(K(\mathbb{Z}/2,2),\mathbb{Z}/2) \xrightarrow{\beta_2} H^6(K(\mathbb{Z}/2,2),\mathbb{Z})$$

$$\mathbb{Z}/4\langle\beta_4 P\gamma_2\rangle \xrightarrow{\cdot 2} \mathbb{Z}/4\langle\beta_4 P\gamma_2\rangle \longrightarrow \mathbb{Z}/2\langle\gamma_2 \operatorname{Sq}^1\gamma_2, \operatorname{Sq}^2\operatorname{Sq}^1\gamma_2\rangle \longrightarrow \mathbb{Z}/2$$

and compute

$$\operatorname{Sq}^1(\gamma_2\operatorname{Sq}^1\gamma_2) = (\operatorname{Sq}^1\gamma_2)^2 \neq 0,$$

$$\operatorname{Sq}^1\operatorname{Sq}^2\operatorname{Sq}^1\gamma_2 = \operatorname{Sq}^3\operatorname{Sq}^1\gamma_2 = (\operatorname{Sq}^1\gamma_2)^2 \neq 0,$$

hence

$$\beta_2(\gamma_2\operatorname{Sq}^1\gamma_2 + \operatorname{Sq}^2\operatorname{Sq}^1\gamma_2) = 0,$$

$$\rho_2\beta_4 P\gamma_2 = \gamma_2\operatorname{Sq}^1\gamma_2 + \operatorname{Sq}^2\operatorname{Sq}^1\gamma_2 \neq 0.$$

To show 6.23, we have to compute $\operatorname{Sq}^2\rho_2\beta_4 P\gamma_2$, so we write

$$\operatorname{Sq}^2(\gamma_2\operatorname{Sq}^1\gamma_2 + \operatorname{Sq}^2\operatorname{Sq}^1\gamma_2) = \operatorname{Sq}^2(\gamma_2\operatorname{Sq}^1\gamma_2) + \operatorname{Sq}^2\operatorname{Sq}^2\operatorname{Sq}^1\gamma_2 =$$

$$= \operatorname{Sq}^2\gamma_2\operatorname{Sq}^1\gamma_2 + \gamma_2\operatorname{Sq}^2\operatorname{Sq}^1\gamma_2 = \gamma_2^2\operatorname{Sq}^1\gamma_2 + \gamma_2\operatorname{Sq}^2\operatorname{Sq}^1\gamma_2 \neq 0.$$

For 6.24 consider the diagram

$$\begin{array}{ccc} & & \mathrm{BSO}(3)_{\leq 4} \\ & \nearrow & \downarrow \\ \mathrm{BSO}(3) & \xrightarrow{w_2} & K(\mathbb{Z}/2,2), \end{array}$$

by naturality and Wu's formula $\operatorname{Sq}^1 w_2 = w_1w_2 + w_3 = w_3$ (w_1 vanishes on BSO(3)), we deduce that $\gamma_2'^2\operatorname{Sq}^1\gamma_2'$ has to be nontrivial in $H^7(\mathrm{BSO}(3)_{\leq 4},\mathbb{Z}/2)$. Similarly, $H^5(\mathrm{BSO}(3)_{\leq 4},\mathbb{Z}/2)$ has to be generated by $\gamma_2'\operatorname{Sq}^1\gamma_2'$.

From the computation of edge homomorphisms and convergence we know that $E^{5,2}$ has to be killed by a class in $E^{0,6}$, denote this class with t.

Remark 6.25. Note that the class t is not defined uniquely, we will fix it later by forcing it to satisfy a certain property.

We conclude that

$$H^6(\mathrm{BSO}(3)_{\leq 4},\mathbb{Z}/2) \simeq \mathbb{Z}/2\langle\gamma_2'^3, (\operatorname{Sq}^1\gamma_2')^2, t\rangle;$$

hence, we are finally ready to compute the second k-invariant of BSO(3). By Lemma 4.12, it is equal to the transgression of the fundamental class in the following Serre exact sequence

$$H^5(\mathrm{BSO}(3)_{\leq 4},\mathbb{Z}/2) \longrightarrow H^5(\mathrm{BSO}(3),\mathbb{Z}/2) \longrightarrow H^5(\mathrm{BSO}(3)_{\geq 5},\mathbb{Z}/2) \xrightarrow{\tau} H^6(\mathrm{BSO}(3)_{\leq 4},\mathbb{Z}/2) \longrightarrow H^6(\mathrm{BSO}(3),\mathbb{Z}/2)$$

$$\mathbb{Z}/2\langle\gamma_2'\operatorname{Sq}^1\gamma_2'\rangle \longrightarrow \mathbb{Z}/2\langle w_2w_3\rangle \xrightarrow{0} \mathbb{Z}/2\langle\gamma_5\rangle \longrightarrow \mathbb{Z}/2\langle\gamma_2'^3, (\operatorname{Sq}^1\gamma_2')^2, t\rangle \longrightarrow \mathbb{Z}/2\langle w_2^3, w_3^2\rangle.$$

As we have already noted γ_2', $\mathrm{Sq}^1\,\gamma_2'$ are mapped to w_2, w_3 respectively. Thus, by exactness, the map

$$\tau : \mathbb{Z}/2\langle \gamma_5 \rangle \to \mathbb{Z}/2\langle \gamma_2'^3, (\mathrm{Sq}^1\,\gamma_2')^2, t\rangle$$

is injective, therefore $\tau\gamma_5$ is a nontrivial class in $H^6(\mathrm{BSO}(3)_{\leq 4}, \mathbb{Z}/2)$. We have enough freedom to define class t with the relation $t = \tau\gamma_5$ (image of τ has to have a t component by the Serre exact sequence above), this property defines the class t uniquely. To sum up, we have proven

Theorem 6.26. *The Postnikov tower of* $\mathrm{BSO}(3)$ *through dimension 5 has the following form*

$$\begin{array}{ccccc} & \nearrow & \mathrm{BSO}(3)_{\leq 4} & \xrightarrow{k_2} & K(\mathbb{Z}/2, 6) \\ & & \downarrow & & \\ \mathrm{BSO}(3) & \xrightarrow{w_2} & K(\mathbb{Z}/2, 2) & \xrightarrow{k_1} & K(\mathbb{Z}, 5), \end{array}$$

where $k_1 = \pm\beta_4 P\gamma_2 \in H^5(K(\mathbb{Z}/2,2),\mathbb{Z})$, $k_2 = t \in \mathbb{Z}/2\langle \gamma_2'^3, (\mathrm{Sq}^1\,\gamma_2')^2, t\rangle \simeq H^6(\mathrm{BSO}(3)_{\leq 4}, \mathbb{Z})$ *and the class* t *is uniquely defined by the property* $k_2 = t$.

Remark 6.27. This result is somewhat unsatisfactory since the class t does not have a geometric meaning. Potentially one could classify 3-bundles over 5-complexes if one understood how to properly interpret this class t geometrically.

The computation we performed quickly bootstraps to the computations of the first k-invariants of $\mathrm{BSO}(4)$, $\mathrm{BSTop}(4)$, and $\mathrm{BTSG}(4)$ by the naturality of k-invariants. For this consider a commutative diagram of Postnikov towers

$$\begin{array}{ccccc} \mathrm{BSO}(3) & \longrightarrow & K(\mathbb{Z}/2,2) & \xrightarrow{k_{1,\mathrm{BSO}(3)}} & K(\mathbb{Z},5) \\ \downarrow & & \downarrow\simeq & & \downarrow \\ \mathrm{BSO}(4) & \longrightarrow & K(\mathbb{Z}/2,2) & \xrightarrow{k_{1,\mathrm{BSO}(4)}} & K(\mathbb{Z}\oplus\mathbb{Z},5) \\ \downarrow & & \downarrow\simeq & & \downarrow \\ \mathrm{BSTop}(4) & \longrightarrow & K(\mathbb{Z}/2,2) & \xrightarrow{k_{1,\mathrm{BSTop}(4)}} & K(\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}/2,5) \\ \downarrow & & \downarrow\simeq & & \downarrow \\ \mathrm{BTSG}(4) & \longrightarrow & K(\mathbb{Z}/2,2) & \xrightarrow{k_{1,\mathrm{BTSG}(4)}} & K(\mathbb{Z}/12\oplus\mathbb{Z},5). \end{array}$$

By Lemma 4.8 the whole diagram is commutative up to homotopy. Also, by Remark 6.20 we know all vertical maps in the middle and the right columns. Putting these facts together we deduce that

$$\begin{aligned} k_{1,\mathrm{BSO}(4)} &= (\pm\beta_4 P\gamma_2, 0), \\ k_{1,\mathrm{BSTop}(4)} &= (\pm\beta_4 P\gamma_2, 0, 0), \\ k_{1,\mathrm{BTSG}(4)} &= (\pm\rho_{12}\beta_4 P\gamma_2, 0). \end{aligned}$$

To compute the second k-invariants, we need to understand the sixth cohomology groups of 4-truncations of these spaces with coefficients in $\mathbb{Z}/2$, thus we turn our attention to this

problem.
From the computation of the first k-invariants we have the following identification of the 4-truncations (recall Lemma 4.8):

$$\begin{aligned} \mathrm{BSO}(4)_{\leq 4} &= \mathrm{BSO}(3)_{\leq 4} \times K(\mathbb{Z}, 4), \\ \mathrm{BSTop}(4)_{\leq 4} &= \mathrm{BSO}(4)_{\leq 4} \times K(\mathbb{Z}/2, 4), \\ \mathrm{BTSG}(4)_{\leq 4} &= F \times K(\mathbb{Z}, 4), \end{aligned} \tag{6.28}$$

where $F = \mathrm{fib}(K(\mathbb{Z}/2, 2) \xrightarrow{\rho_{12}\beta_4 P\gamma_2} K(\mathbb{Z}/12, 5))$. Therefore, we obtain the following commutative diagram of Postnikov towers (also by Lemma 4.8):

$$\begin{array}{ccccccccc} \mathrm{BSO}(3) & \longrightarrow & \mathrm{BSO}(3)_{\leq 4} & \xrightarrow{=} & \mathrm{BSO}(3)_{\leq 4} & \longrightarrow & K(\mathbb{Z}/2, 2) & \longrightarrow & K(\mathbb{Z}, 5) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathrm{BSO}(4) & \longrightarrow & \mathrm{BSO}(4)_{\leq 4} & \xrightarrow{=} & \mathrm{BSO}(3)_{\leq 4} \times K(\mathbb{Z}, 4) & \longrightarrow & K(\mathbb{Z}/2, 2) & \longrightarrow & K(\mathbb{Z} \oplus \mathbb{Z}, 5) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathrm{BSTop}(4) & \longrightarrow & \mathrm{BSTop}(4)_{\leq 4} & \xrightarrow{=} & \mathrm{BSO}(4)_{\leq 4} \times K(\mathbb{Z}/2, 4) & \longrightarrow & K(\mathbb{Z}/2, 2) & \longrightarrow & K(\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2, 5) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathrm{BTSG}(4) & \longrightarrow & \mathrm{BTSG}(4)_{\leq 4} & \xrightarrow{=} & F \times K(\mathbb{Z}, 4) & \longrightarrow & K(\mathbb{Z}/2, 2) & \longrightarrow & K(\mathbb{Z}/12 \oplus \mathbb{Z}, 5), \end{array} \tag{6.29}$$

where the maps in the middle column are induced by the maps in the right part of the diagram by the functoriality of the homotopy fiber. Using the Künneth formula, we can compute ranks of the mod 2 cohomology of these 4-truncations. Furthermore, we will also describe generators of these groups in more geometric terms using naturality. We still need to compute cohomology groups of F first. To do this, we apply Serre spectral sequence to the fiber sequence

$$F \longrightarrow K(\mathbb{Z}/2, 2) \xrightarrow{\rho_{12}\beta_4 P\gamma_2} K(\mathbb{Z}/12, 5).$$

We copy a table of the relevant cohomology groups of Eilenberg-Maclane spaces from Appendix A and provide this spectral sequence, followed by an explanation of how to fill it.

$H^i(K(\mathbb{Z}/2, n), \mathbb{Z}/2)$	$i = 3$	4	5	6	7
$n = 2$	$\langle \mathrm{Sq}^1 \gamma_2 \rangle$	$\langle \gamma_2^2 \rangle$	$\langle \gamma_2 \mathrm{Sq}^1 \gamma_2, \mathrm{Sq}^2 \mathrm{Sq}^1 \gamma_2 \rangle$	$\langle \gamma_2^3, (\mathrm{Sq}^1 \gamma_2)^2 \rangle$	$\langle \gamma_2 \mathrm{Sq}^2 \mathrm{Sq}^1 \gamma_2,$ $\gamma_2^2 \mathrm{Sq}^1 \gamma_2 \rangle$
$H^i(K(\mathbb{Z}/12, n), \mathbb{Z}/2)$	$i = 3$	4	5	6	7
$n = 5$	0	0	$\langle \gamma_5 \rangle$	$\langle \zeta_5 \rangle$	$\langle \mathrm{Sq}^2 \gamma_5 \rangle$

q	0	1	2	3	4	5	6	7
6	$\gamma_2'^3, (\mathrm{Sq}^1 \gamma_2')^2, t$	0						
5	$\gamma_2' \mathrm{Sq}^1 \gamma_2', k$	0	0					
4	$\gamma_2'^2$	0	0	0				
3	$\mathrm{Sq}^1 \gamma_2'$	0	0	0	0			
2	γ_2'	0	0	0	0	$\gamma_2'\gamma_5$		
1	0	0	0	0	0	0	0	
0	$\mathbb{Z}/2$	0	0	0	0	γ_5	ζ_5	$\mathrm{Sq}^2 \gamma_5$

(6.30)

$$H^p(K(\mathbb{Z}/12, 5), H^q(F, \mathbb{Z}/2)) \implies H^{p+q}(K(\mathbb{Z}/2, 2), \mathbb{Z}/2).$$

As in the case of $\mathrm{BSO}(3)_{\leq 4}$, we conclude that

$$H^2(F, \mathbb{Z}/2) = \mathbb{Z}/2\langle \gamma_2' \rangle,$$
$$H^3(F, \mathbb{Z}/2) = \mathbb{Z}/2\langle \mathrm{Sq}^1 \gamma_2' \rangle,$$
$$H^4(F, \mathbb{Z}/2) = \mathbb{Z}/2\langle \gamma_2'^2 \rangle.$$

Furthermore, we have a commutative diagram

$$\begin{array}{ccc} & \overset{\alpha}{\nearrow} & F \\ & & \downarrow \\ \mathrm{BSO}(3) & \xrightarrow{w_2} & K(\mathbb{Z}/2, 2), \end{array}$$

where α is given by the composition

$$\mathrm{BSO}(3) \to \mathrm{BTSG}(4) \to \mathrm{BTSG}(4)_{\leq 4} = F \times K(\mathbb{Z}, 4) \xrightarrow{\mathrm{pr}_1} F.$$

By naturality and Wu's formula we again deduce that $\gamma_2' \mathrm{Sq}^1 \gamma_2'$, $\gamma_2'^3$, $(\mathrm{Sq}^1 \gamma_2')^2$, and $\gamma_2'^2 \mathrm{Sq}^1 \gamma_2'$ are nontrivial in $H^*(F, \mathbb{Z}/2)$. Furthermore, the same computation as for $\mathrm{BSO}(3)_{\leq 4}$ shows that the edge homomorphisms

$$H^5(K(\mathbb{Z}/12, 5), \mathbb{Z}/2) \to H^5(K(\mathbb{Z}/2, 2), \mathbb{Z}/2),$$

$$H^7(K(\mathbb{Z}/12,5),\mathbb{Z}/2) \to H^7(K(\mathbb{Z}/2,2),\mathbb{Z}/2),$$
$$H^7(K(\mathbb{Z}/2,2),\mathbb{Z}/2) \to H^7(F,\mathbb{Z}/2)$$

are nontrivial. Thus, from convergence, we conclude that there have to be classes $k \in E^{0,5}$, $t \in E^{0,6}$ which will kill groups $E^{6,0}, E^{5,2}$ respectively.

Remark 6.31. Notice that there is a map of spectral sequences from 6.30 to 6.21, an examination of this map shows that the homomorphism

$$H^6(F,\mathbb{Z}/2) \to H^6(\mathrm{BSO}(3)_{\le 4},\mathbb{Z}/2)$$

is an isomorphism. From now on we identify the class $t \in H^6(F,\mathbb{Z}/2)$ with the pullback of the class $t \in H^6(\mathrm{BSO}(3)_{\le 4},\mathbb{Z}/2)$ along this isomorphism.

Now by applying the Künneth formula to the splittings 6.28 we derive the following table

	$i = 1$	2	3
$H^i(\mathrm{BSO}(4)_{\le 4},\mathbb{Z}/2)$	0	$\mathbb{Z}/2\langle\gamma_2'\rangle$	$\mathbb{Z}/2\langle\mathrm{Sq}^1\gamma_2'\rangle$
$H^i(\mathrm{BSTop}(4)_{\le 4},\mathbb{Z}/2)$	0	$\mathbb{Z}/2\langle\gamma_2'\rangle$	$\mathbb{Z}/2\langle\mathrm{Sq}^1\gamma_2'\rangle$
$H^i(\mathrm{BTSG}(4)_{\le 4},\mathbb{Z}/2)$	0	$\mathbb{Z}/2\langle\gamma_2'\rangle$	$\mathbb{Z}/2\langle\mathrm{Sq}^1\gamma_2'\rangle$
	$i = 4$	5	6
$H^i(\mathrm{BSO}(4)_{\le 4},\mathbb{Z}/2)$	$\mathbb{Z}/2\langle\gamma_2'^2,\rho_2\iota_4'\rangle$	$\mathbb{Z}/2\langle\gamma_2'\,\mathrm{Sq}^1\gamma_2'\rangle$	$\mathbb{Z}/2\langle\gamma_2'^3,(\mathrm{Sq}^1\gamma_2')^2,\gamma_2'\rho_2\iota_4',$ $\mathrm{Sq}^2\rho_2\iota_4',t\rangle$
$H^i(\mathrm{BSTop}(4)_{\le 4},\mathbb{Z}/2)$	$\mathbb{Z}/2\langle\gamma_2'^2,\rho_2\iota_4',\gamma_4'\rangle$	$\mathbb{Z}/2\langle\gamma_2'\,\mathrm{Sq}^1\gamma_2',\mathrm{Sq}^1\gamma_4'\rangle$	$\mathbb{Z}/2\langle\gamma_2'^3,(\mathrm{Sq}^1\gamma_2')^2,\gamma_2'\rho_2\iota_4',$ $\mathrm{Sq}^2\rho_2\iota_4',\gamma_2'\gamma_4',\mathrm{Sq}^2\gamma_4',t\rangle$
$H^i(\mathrm{BTSG}(4)_{\le 4},\mathbb{Z}/2)$	$\mathbb{Z}/2\langle\gamma_2'^2,\rho_2\iota_4'\rangle$	$\mathbb{Z}/2\langle\gamma_2'\,\mathrm{Sq}^1\gamma_2',k\rangle$	$\mathbb{Z}/2\langle\gamma_2'^3,(\mathrm{Sq}^1\gamma_2')^2,\gamma_2'\rho_2\iota_4',$ $\mathrm{Sq}^2\rho_2\iota_4',t\rangle$

Where the listed generators come from the following groups under the Künneth formula:

$$\gamma_2' \in H^2(\mathrm{BSO}(3)_{\le 4},\mathbb{Z}/2),$$
$$\rho_2\iota_4 \in H^4(K(\mathbb{Z},4),\mathbb{Z}/2),$$
$$\gamma_4 \in H^4(K(\mathbb{Z}/2,4),\mathbb{Z}/2).$$

Remark 6.32. The idea is that the second k-invariant is an element in $H^6(\mathrm{BSTop}(4),\mathbb{Z}/2\oplus\mathbb{Z}/2)$, so it is given by a pair of classes in $H^6(\mathrm{BSTop}(4),\mathbb{Z}/2)$ each of which we will show to be a linear combination of $a\gamma_2'\rho_2\iota_4' + t$ and $\gamma_2'\rho_2\iota_4' + \mathrm{Sq}^2\rho_2\iota_4'$, for some $a \in \mathbb{Z}/2$. It is important that these classes appear in $\mathrm{BSTop}(4)_{\le 4}$ by pulling them back along the projection

$$\mathrm{BSTop}(4)_{\le 4} = \mathrm{BSO}(4)_{\le 4} \times K(\mathbb{Z}/2,4) \xrightarrow{\mathrm{pr}_1} \mathrm{BSO}(4)_{\le 4}$$

and this will tell us that the second k-invariant of $\mathrm{BSTop}(4)$ is the same as that of $\mathrm{BSO}(4)$.

We know the induced maps between these cohomology groups up to degree 4 because we know behavior of the maps in the middle column of Diagram 6.29. Specifically, generators with the same names are mapped to each other, except for the class t, which we will discuss soon.

For this, we first identify to which classes all these generators correspond in their geometric counterparts $\mathrm{BSO}(4), \mathrm{BSTop}(4), \mathrm{BTSG}(4)$.
Firstly, we already know that γ_2' and $\mathrm{Sq}^1\,\gamma_2'$ are mapped to w_2, w_3 in $\mathrm{BTSG}(4), \mathrm{BSTop}(4), \mathrm{BSO}(4)$. Denote the fundamental class of $K(\mathbb{Z},4)$ in $\mathrm{BSO}(4)_{\leq 4} = \mathrm{BSO}(3)_{\leq 4} \times K(\mathbb{Z},4)$ by $\iota_4' \in H^4(\mathrm{BSO}(4)_{\leq 4}, \mathbb{Z})$. Then we claim that it is mapped to e in $\mathrm{BSO}(4)$. For this consider the following composition

$$\begin{array}{ccccc} S^4 & \xrightarrow{\xi} & \mathrm{BSO}(4) & \longrightarrow & \mathrm{BSO}(3)_{\leq 4} \times K(\mathbb{Z},4) \\ & \searrow & & & \downarrow \mathrm{pr}_2 \\ & & & & K(\mathbb{Z},4), \end{array} \tag{6.33}$$

where ξ corresponds to a generator of $\pi_4\,\mathrm{BSO}(3)$, we know that the whole composition has to be to a linear combination $ap_1(\xi) + be(\xi)$ for some $a, b \in \mathbb{Z}$. On the other hand in π_4 we know that the map has to be $\mathbb{Z} \xrightarrow{0} \mathbb{Z}$, this means that it is trivial on $H^4(-,\mathbb{Z})$ and thus $ap_1(\xi) + be(\xi) = 0$. Moreover, we know that $p_1(\xi) = \pm 4$ and $e(\xi) = 0$ (see, for example, [Mil56]), hence $a = 0$. Now we consider the same composition but with ξ representing the generator of the second infinite cyclic summand of $\pi_4\,\mathrm{BSO}(4)$, namely the one given by the section from Lemma 6.2. Then by the similar reasoning $H^4(K(\mathbb{Z},4),\mathbb{Z}) \to H^4(S^4,\mathbb{Z})$ is the identity and $e(\xi) = 1$ by [Mil56]. Therefore, $b = 1$ and ι_4' is mapped to e. This also implies that $\iota_4' \in H^4(\mathrm{BSO}(4)_{\leq 4} \times K(\mathbb{Z}/2,4),\mathbb{Z})$ is mapped to $e \in H^4(\mathrm{BSTop}(4),\mathbb{Z})$ and $\rho_2\iota_4' \in H^4(F \times K(\mathbb{Z},4),\mathbb{Z}/2)$ is mapped to $w_4 \in H^4(\mathrm{BTSG}(4),\mathbb{Z}/2)$ because the maps

$$H^4(\mathrm{BSTop}(4),\mathbb{Z}) \to H^4(\mathrm{BSO}(4),\mathbb{Z}),$$
$$H^4(\mathrm{BTSG}(4),\mathbb{Z}/2) \to H^4(\mathrm{BSO}(4),\mathbb{Z}/2)$$

are isomorphisms. Indeed, this follows from the fact that the same maps are isomorphisms on 4-truncations (6.28) by the Künneth formula.
Finally, we claim that $\gamma_4' \in H^4(\mathrm{BSO}(4)_{\leq 4} \times K(\mathbb{Z}/2,4),\mathbb{Z}/2)$ is mapped to $\mathrm{ks} \in H^4(\mathrm{BSTop}(4),\mathbb{Z}/2)$. For this consider the composition

$$\begin{array}{ccccc} S^4 & \xrightarrow{\xi} & \mathrm{BSTop}(4) & \longrightarrow & \mathrm{BSO}(4)_{\leq 4} \times K(\mathbb{Z}/2,4), \\ & \searrow & & & \downarrow \mathrm{pr}_2 \\ & & & & K(\mathbb{Z}/2,4) \end{array}$$

where ξ corresponds to the 2 torsion class in $\pi_4\,\mathrm{BSTop}(4)$. We know that the composition corresponds to a $\mathbb{Z}/2$ linear combination $aw_2^2(\xi) + bw_4(\xi) + c\,\mathrm{ks}(\xi)$, it is an isomorphism in $H^4(-,\mathbb{Z}/2)$, and $\mathrm{ks}(\xi) = 1$. We also know that γ_4' maps to zero in $H^4\,\mathrm{BSO}(4)_{\leq 4}$ and to $aw_2^2 + bw_4 \in H^4\,\mathrm{BSO}(4)$ by the commutativity of

$$\begin{array}{ccccc} \mathrm{BSO}(4) & \longrightarrow & \mathrm{BSO}(4)_{\leq 4} & & \\ \downarrow & & \downarrow & \overset{\simeq *}{\searrow} & \\ \mathrm{BSTop}(4) & \longrightarrow & \mathrm{BSO}(4)_{\leq 4} \times K(\mathbb{Z}/2,4) & \xrightarrow[\mathrm{pr}_2]{} & K(\mathbb{Z}/2,4) \end{array}$$

and because ks maps to 0 in BSO(4) by the definition of the Kirby-Siebenmann class. Therefore, $a = b = 0$ and $c = 1$.
Now we deal with the class t. Recall that it was defined as the second nontrivial k-invariant of BSO(3) and it appears in $H^6(\mathrm{BSO}(4)_{\leq 4}, \mathbb{Z}/2)$, $H^6(\mathrm{BSTop}(4)_{\leq 4}, \mathbb{Z}/2)$ by pulling back from $H^6(\mathrm{BSO}(3)_{\leq 4}, \mathbb{Z}/2)$ along the projections

$$\mathrm{BSO}(4)_{\leq 4} = \mathrm{BSO}(3)_{\leq 4} \times K(\mathbb{Z}, 4) \xrightarrow{\mathrm{pr}_1} \mathrm{BSO}(3)_{\leq 4},$$

$$\mathrm{BSTop}(4)_{\leq 4} = \mathrm{BSO}(4)_{\leq 4} \times K(\mathbb{Z}/2, 4) \xrightarrow{\mathrm{pr}_1} \mathrm{BSO}(4)_{\leq 4},$$

and in $\mathrm{BTSG}(4)_{\leq 4}$ via the isomorphism from Remark 6.31

$$H^6(F, \mathbb{Z}/2) \xrightarrow{\simeq} H^6(\mathrm{BSO}(3)_{\leq 4}, \mathbb{Z}/2).$$

The induced map

$$H^6(F \times K(\mathbb{Z}, 4), \mathbb{Z}/2) \to H^6(\mathrm{BSO}(3)_{\leq 4} \times K(\mathbb{Z}, 4), \mathbb{Z}/2)$$

is an isomorphism because $H^6(F) \xrightarrow{\simeq} H^6(\mathrm{BSO}(3)_{\leq 4})$ is an isomorphism and the classes $\gamma_2' \rho_2 \iota_4'$, $\mathrm{Sq}^2 \rho_2 \iota_4'$ are mapped to the same classes by the discussion above. Therefore, the class t is also mapped to the same class via this map. Lastly, we will prove that t is also mapped to t under the map

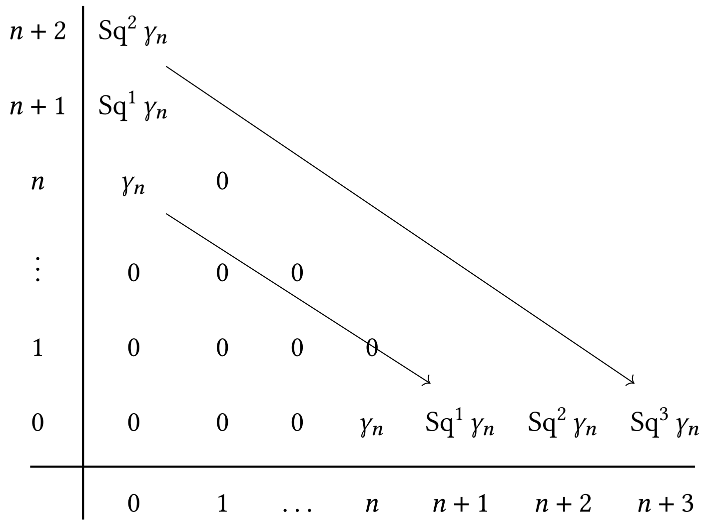
$$H^6(F \times K(\mathbb{Z}, 4), \mathbb{Z}/2) \to H^6(\mathrm{BSO}(4)_{\leq 4} \times K(\mathbb{Z}/2, 4), \mathbb{Z}/2).$$

We already know that it can only be mapped to a linear combination $t + a \,\mathrm{Sq}^2 \gamma_4' + b \gamma_2' \gamma_4'$ and we will show that $a = b = 0$. Begin with

Lemma 6.34. *The map* $K(\mathbb{Z}/2, n) \xrightarrow{\cdot 6} K(\mathbb{Z}/12, n)$ *induces zero on* $H^n(-, \mathbb{Z}/2)$ *and* $H^{n+2}(-, \mathbb{Z}/2)$ *for* $n > 3$.

Proof. Consider the Serre spectral sequence of the fibration

$$K(\mathbb{Z}/2, n) \xrightarrow{\cdot 6} K(\mathbb{Z}/12, n) \to K(\mathbb{Z}/6, n)$$



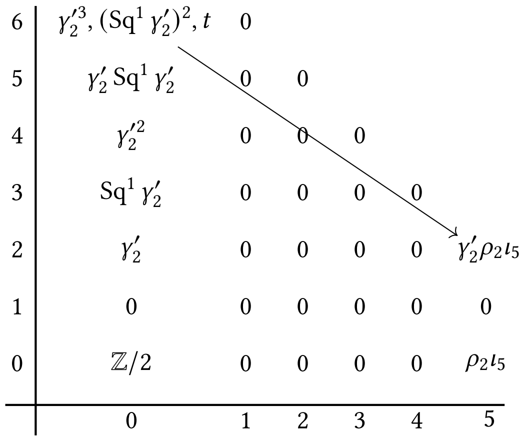
$$H^p(K(\mathbb{Z}/6,n),H^q(K(\mathbb{Z}/2,n),\mathbb{Z}/2)) \implies H^{p+q}(K(\mathbb{Z}/12,n),\mathbb{Z}/2).$$

The statement follows by an inspection of the edge homomorphisms. □

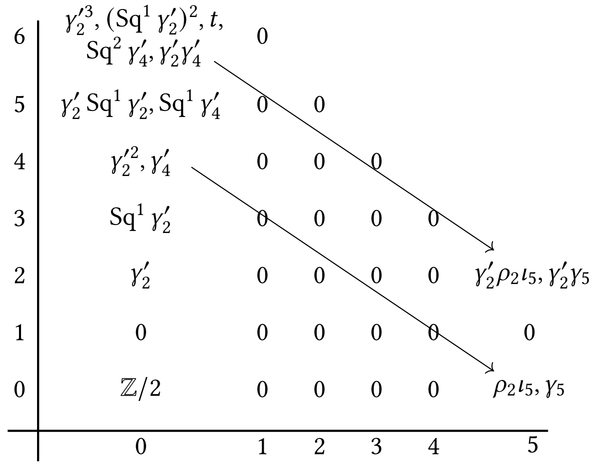
Firstly, we consider the following maps of fiber sequences

$$\begin{array}{ccccc}
\mathrm{BSO}(3)_{\leq 4} & \longrightarrow & K(\mathbb{Z}/2,2) & \xrightarrow{\pm\beta_4 P\gamma_2} & K(\mathbb{Z},5) \\
\uparrow{\scriptstyle \mathrm{pr}_1} & & \simeq\uparrow & & {\scriptstyle \mathrm{pr}_1}\uparrow \\
\mathrm{BSO}(3)_{\leq 4}\times K(\mathbb{Z}/2,4) & \longrightarrow & K(\mathbb{Z}/2,2) & \xrightarrow{(\pm\beta_4 P\gamma_2,0)} & K(\mathbb{Z}\oplus\mathbb{Z}/2,5) \\
\downarrow & & \downarrow\simeq & & \downarrow f \\
F & \longrightarrow & K(\mathbb{Z}/2,2) & \xrightarrow{\pm\rho_{12}\beta_4 P\gamma_2} & K(\mathbb{Z}/12,5),
\end{array} \tag{6.35}$$

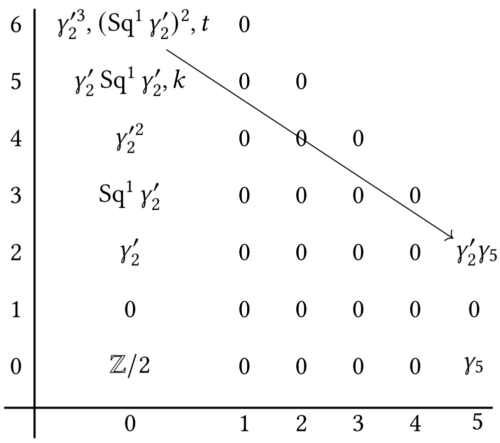
where f is given by $f(x,y)=x+6y$ (recall Remark 6.20). These fiber sequences induce Serre spectral sequences and maps between them:



A1: $H^p(K(\mathbb{Z},5), H^q(\mathrm{BSO}(3)_{\leq 4}, \mathbb{Z}/2)) \implies H^{p+q}(K(\mathbb{Z}/2,2),\mathbb{Z}/2)$.



B1: $H^p(K(\mathbb{Z}\oplus\mathbb{Z}/2,5), H^q(\mathrm{BSO}(3)_{\leq 4}\times K(\mathbb{Z}/2,4), \mathbb{Z}/2)) \implies H^{p+q}(K(\mathbb{Z}/2,2),\mathbb{Z}/2)$.



C1: $H^p(K(\mathbb{Z}/12,5), H^q(F,\mathbb{Z}/2)) \implies H^{p+q}(K(\mathbb{Z}/2,2),\mathbb{Z}/2)$.

We have a map $A1 \to B1$, by comparing the differentials $d : E^{0,6} \to E^{5,2}$ in these spectral sequences, we deduce that $d(t) = \gamma_2'\rho_2\iota_5$ in B1. We also have a map $C1 \to B1$, by Lemma 6.34 we know that γ_5 in C1 is mapped to $\rho_2\iota_5$ in B1. By comparing the differentials in B1 and C1 we have the following commutative square

$$\begin{array}{ccc} t + a\,\mathrm{Sq}^2\gamma_4' + b\gamma_2'\gamma_4' & \xmapsto{d_{B1}} & \gamma_2'\rho_2\iota_5 + b\gamma_2'\gamma_5 = \gamma_2'\rho_2\iota_5 \\ \uparrow & & \uparrow \\ t & \xmapsto{d_{C1}} & \gamma_2'\gamma_5 \end{array}$$

because $d_{B1}(\mathrm{Sq}^2\gamma_4') = 0$ ($\mathrm{Sq}^2\gamma_4'$ maps to $\mathrm{Sq}^2\gamma_4$ via the edge homomorphism) and $d_{B1}(\gamma_2'\gamma_4') = \gamma_2'\gamma_5$ (by multiplicativity); thus $b = 0$. Secondly, an inspection of the spectral sequence of the fiber sequence

$$K(\mathbb{Z}/12,4) \longrightarrow F \longrightarrow K(\mathbb{Z}/2,2)$$

6	$\mathrm{Sq}^2\gamma_4$							
5	ζ_4	0						
4	γ_4	0	$\gamma_4\gamma_2$					
3	0	0	0	0				
2	0	0	0	0	0			
1	0	0	0	0	0	0		
0	$\mathbb{Z}/2$	0	γ_2	$\mathrm{Sq}^1\gamma_2$	γ_2^2	$\mathrm{Sq}^2\mathrm{Sq}^1\gamma_2$, $\gamma_2\mathrm{Sq}^1\gamma_2$	γ_2^3, $(\mathrm{Sq}^1\gamma_2)^2$	γ_2^4, $\gamma_2(\mathrm{Sq}^1\gamma_2)^2$, $\mathrm{Sq}^1\gamma_2\mathrm{Sq}^2\mathrm{Sq}^1\gamma_2$
	0	1	2	3	4	5	6	7

$$H^p(K(\mathbb{Z}/2,2), H^q(K(\mathbb{Z}/12,4),\mathbb{Z}/2)) \implies H^{p+q}(F,\mathbb{Z}/2)$$

implies that the edge homomorphism

$$H^6(F,\mathbb{Z}/2) \to H^6(K(\mathbb{Z}/12,4),\mathbb{Z}/2)$$

maps t to $\mathrm{Sq}^2\gamma_4$. Consider the following commutative square given by rotating two bottom fiber sequences in 6.35 once

$$\begin{array}{ccc} K(\mathbb{Z}\oplus\mathbb{Z}/2,4) & \longrightarrow & \mathrm{BSO}(3)_{\leq 4}\times K(\mathbb{Z}/2,4) \\ \big\downarrow{\scriptstyle \Omega f} & & \big\downarrow \\ K(\mathbb{Z}/12,4) & \longrightarrow & F, \end{array}$$

by Lemma 6.34 we know the map induced in cohomology by the left vertical arrow; namely, we have the following commutative square in sixth cohomology

$$\begin{array}{ccc} \mathrm{Sq}^2\rho_2\iota_4 & \longleftarrow\!\shortmid & t + a\,\mathrm{Sq}^2\gamma_4' \\ \big\uparrow & & \big\uparrow \\ \mathrm{Sq}^2\gamma_4 & \longleftarrow\!\shortmid & t. \end{array}$$

Thus, we conclude that $a = 0$, because otherwise $\mathrm{Sq}^2\gamma_4$ would appear in the top left corner. To sum up, we proved

Proposition 6.36. *There is a commutative diagram with vertical arrows being isomorphisms*

$$\begin{array}{ccccc}
\mathbb{Z}/2\langle \gamma_2'^3, (\mathrm{Sq}^1\gamma_2')^2, \mathrm{Sq}^2\rho_2\iota_4', & \longrightarrow & \mathbb{Z}/2\langle \gamma_2'^3, (\mathrm{Sq}^1\gamma_2')^2, \mathrm{Sq}^2\rho_2\iota_4', & \longrightarrow & \mathbb{Z}/2\langle \gamma_2'^3, (\mathrm{Sq}^1\gamma_2')^2, \mathrm{Sq}^2\rho_2\iota_4', \\
\gamma_2'\rho_2\iota_4', t\rangle & & \gamma_2'\rho_2\iota_4', \gamma_2'\gamma_4', \mathrm{Sq}^2\gamma_4', t\rangle & & \gamma_2'\rho_2\iota_4', t\rangle \\
\downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq \\
H^6(\mathrm{BTSG}(4)_{\leq 4}, \mathbb{Z}/2) & \longrightarrow & H^6(\mathrm{BSTop}(4)_{\leq 4}, \mathbb{Z}/2) & \longrightarrow & H^6(\mathrm{BSO}(4)_{\leq 4}, \mathbb{Z}/2)
\end{array}$$

and the top horizontal arrows map classes with the same names to each other.

Finally, we are ready to prove Theorem 6.1. For this consider the diagram of Postnikov towers

$$\begin{array}{ccccccc}
\mathrm{BSO}(4)_{\geq 5} & \longrightarrow & \mathrm{BSO}(4) & \longrightarrow & \mathrm{BSO}(4)_{\leq 4} & \longrightarrow & K(\mathbb{Z}/2\oplus\mathbb{Z}/2, 6) \\
\downarrow & & \downarrow & & \downarrow & & \simeq\downarrow \\
\mathrm{BSTop}(4)_{\geq 5} & \longrightarrow & \mathrm{BSTop}(4) & \longrightarrow & \mathrm{BSTop}(4)_{\leq 4} & \longrightarrow & K(\mathbb{Z}/2\oplus\mathbb{Z}/2, 6) \\
\downarrow & & \downarrow & & \downarrow & & \simeq\downarrow \\
\mathrm{BSG}(4)_{\geq 5} & \longrightarrow & \mathrm{BSG}(4) & \longrightarrow & \mathrm{BSG}(4)_{\leq 4} & \longrightarrow & K(\mathbb{Z}/2\oplus\mathbb{Z}/2, 6),
\end{array}$$

we apply Lemma 4.12 to these Postnikov towers and investigate the corresponding diagram of the Serre exact sequences

$$\begin{array}{ccccc}
H^5(\mathrm{BSO}(4)_{\geq 5}, \mathbb{Z}/2\oplus\mathbb{Z}/2) & \xrightarrow{\tau_{\mathrm{BSO}}} & H^6(\mathrm{BSO}(4)_{\leq 4}, \mathbb{Z}/2\oplus\mathbb{Z}/2) & \longrightarrow & H^6(\mathrm{BSO}(4), \mathbb{Z}/2\oplus\mathbb{Z}/2) \\
\mathrm{Id}\uparrow & & \uparrow & & \uparrow \\
H^5(\mathrm{BSTop}(4)_{\geq 5}, \mathbb{Z}/2\oplus\mathbb{Z}/2) & \xrightarrow{\tau_{\mathrm{BSTop}}} & H^6(\mathrm{BSTop}(4)_{\leq 4}, \mathbb{Z}/2\oplus\mathbb{Z}/2) & \longrightarrow & H^6(\mathrm{BSTop}(4), \mathbb{Z}/2\oplus\mathbb{Z}/2) \\
\mathrm{Id}\uparrow & & \alpha\uparrow & & \uparrow \\
H^5(\mathrm{BSG}(4)_{\geq 5}, \mathbb{Z}/2\oplus\mathbb{Z}/2) & \xrightarrow{\tau_{\mathrm{BSG}}} & H^6(\mathrm{BSG}(4)_{\leq 4}, \mathbb{Z}/2\oplus\mathbb{Z}/2) & \longrightarrow & H^6(\mathrm{BSG}(4), \mathbb{Z}/2\oplus\mathbb{Z}/2).
\end{array}$$

The whole diagram splits as the direct sum of two copies of the same diagram, but with $\mathbb{Z}/2$ coefficients, since the corresponding Serre spectral sequences also split in the same manner. Moreover, from Remark 6.20, we know that the both leftmost vertical arrows are the identity and we know the middle vertical maps from Proposition 6.36. By commutativity of the diagram, we have $k_{2,\mathrm{BSTop}(4)} = \tau_{\mathrm{BSTop}}(\gamma_5) \in H^6(\mathrm{BSTop}(4)_{\leq 4}, \mathbb{Z}/2\oplus\mathbb{Z}/2)$ is equal to $\alpha(\tau_{\mathrm{BSG}}(\gamma_5))$. By Proposition 6.36, this implies that $k_{2,\mathrm{BSTop}(4)}$ is the same as $k_{2,\mathrm{BSG}(4)} = k_{2,\mathrm{BSO}(4)}$.

Remark 6.37. Note that we do not know exactly what this k-invariant is, the best we can say is that it is a linear combination of $a\gamma_2'\rho_2\iota_4' + t$ and $\gamma_2'\rho_2\iota_4' + \mathrm{Sq}^2\iota_4'$ for some $a \in \mathbb{Z}/2$ in both copies of $\mathbb{Z}/2$ coefficients in $H^6(\mathrm{BSTop}(4)_{\leq 4}, \mathbb{Z}/2\oplus\mathbb{Z}/2)$. The first class appears because t is the k-invariant of BSO(3) and the second class appears because it is mapped to $w_2w_4 + \mathrm{Sq}^2 w_4$ in $H^6(\mathrm{BSTop}(4), \mathbb{Z}/2)$ which is zero by Wu's formula.

We have finally proved Theorem 6.1 because $k_{2,\mathrm{BSTop}(4)} \in H^6(\mathrm{BSO}(4)_{\leq 4}\times K(\mathbb{Z}/2,4), \mathbb{Z}/2\oplus\mathbb{Z}/2)$ comes from $\mathrm{BSO}(4)_{\leq 4}$ under the projection map, therefore

$$\mathrm{BSTop}(4)_{\leq 5} \cong \mathrm{BSO}(4)_{\leq 5}\times K(\mathbb{Z}/2,4) = \mathrm{fib}(\mathrm{BSO}(4)_{\leq 4}\times K(\mathbb{Z}/2,4) \xrightarrow{k_{2,\mathrm{BSO}(4)}\circ \mathrm{pr}_1} K(\mathbb{Z}/2\oplus\mathbb{Z}/2, 6)).$$

We have already discussed why the $K(\mathbb{Z}/2,4)$ factor corresponds to the Kirby-Siebenmann class and the composition

$$\mathrm{BSO}(4) \to \mathrm{BSTop}(4) \to \mathrm{BSTop}(4)_{\leq 5} \cong \mathrm{BSO}(4)_{\leq 5} \times K(\mathbb{Z}/2,4) \xrightarrow{\mathrm{pr}_1} \mathrm{BSO}(4)_{\leq 5}$$

is 6-connected because the following diagram commutes by Lemma 4.8

$$\begin{array}{ccc} \mathrm{BSO}(4) & \longrightarrow & \mathrm{BSO}(4)_{\leq 5} \\ \downarrow & & \downarrow{\scriptstyle i_1} \\ \mathrm{BSTop}(4) & \longrightarrow & \mathrm{BSO}(4)_{\leq 5} \times K(\mathbb{Z}/2,4). \end{array}$$

7 Classification of vector bundles and microbundles

In this section, we recall the theorem of Dold and Whitney on the classification of oriented vector bundles over CW complexes of dimension 4 (Theorem 7.1) and prove a similar statement for microbundles (Theorem 7.2) using computations from Section 6. It is possible to deduce Theorem 7.2 from Theorem 7.1 and Theorem 6.1 directly, but for the completeness of the exposition we prove it without using the theorem of Dold and Whitney. We use an approach similar to [ČV93]; we consider a map from BSTop(4) to a product of Eilenberg-Maclane spaces given by characteristic classes and investigate its Moore-Postnikov tower.

Theorem 7.1 (Dold-Whitney [DW59]). *Let X be a CW complex of dimension 4 with no 2-torsion in $H^4(X,\mathbb{Z})$. Then the maps:*

$$\mathrm{Vect}_3^+(X) \xrightarrow{(w_2,p_1)} H^2(X,\mathbb{Z}/2)\times H^4(X,\mathbb{Z}),$$

$$\mathrm{Vect}_4^+(X) \xrightarrow{(w_2,e,p_1)} H^2(X,\mathbb{Z}/2)\times H^4(X,\mathbb{Z})\times H^4(X,\mathbb{Z}),$$

$$\mathrm{Vect}_m^+(X) \xrightarrow{(w_2,w_4,p_1)} H^2(X,\mathbb{Z}/2)\times H^4(X,\mathbb{Z}/2)\times H^4(X,\mathbb{Z}),\ \textit{for } m>4;$$

are injective and their images consist of the following sets of tuples respectively:

$$\{(a,c):\rho_4 c = Pa\},$$

$$\{(a,b,c):\rho_4 c = Pa + 2\rho_4 b\},$$

$$\{(a,b,c):\rho_4 c = Pa + \iota_* b\},$$

where $P: H^2(-,\mathbb{Z}/2)\to H^4(-,\mathbb{Z}/4)$ is the Pontryagin square and $\iota_*: H^4(-,\mathbb{Z}/2)\to H^4(-,\mathbb{Z}/4)$ is the multiplication by 2.

Now we will formulate a similar statement for microbundles. Firstly, notice that the theorem uses the first Pontryagin class which has a slightly different definition for microbundles 5.10. Secondly, we have to take into account the new Kirby-Siebenmann class $\mathrm{ks}\in H^4(\mathrm{BSTop}(4),\mathbb{Z}/2)$.

Theorem 7.2. *Let X be a topological 4-manifold or a locally finite 4-dimensional simplicial complex with no 2-torsion in $H^4(X,\mathbb{Z})$. Then the maps:*

$$\mathrm{Mic}_3^+(X) \xrightarrow{(w_2,\tilde{p}_1)} H^2(X,\mathbb{Z}/2)\times H^4(X,\mathbb{Z}),$$

$$\mathrm{Mic}_4^+(X) \xrightarrow{(w_2,e,\tilde{p}_1,\mathrm{ks})} H^2(X,\mathbb{Z}/2)\times H^4(X,\mathbb{Z})\times H^4(X,\mathbb{Z})\times H^4(X,\mathbb{Z}/2),$$

$$\mathrm{Mic}_m^+(X) \xrightarrow{(w_2,w_4,\tilde{p}_1,\mathrm{ks})} H^2(X,\mathbb{Z}/2)\times H^4(X,\mathbb{Z}/2)\times H^4(X,\mathbb{Z})\times H^4(X,\mathbb{Z}/2),\ \textit{for } m>4;$$

are injective and their images consist of the following sets of tuples respectively:

$$\{(a,c):\rho_4 c = Pa\},$$

$$\{(a,b,c,d):\rho_4 c = Pa + 2\rho_4 b\},$$

$$\{(a,b,c,d):\rho_4 c = Pa + \iota_* b\},$$

where $P: H^2(-,\mathbb{Z}/2)\to H^4(-,\mathbb{Z}/4)$ is the Pontryagin square and $\iota_*: H^4(-,\mathbb{Z}/2)\to H^4(-,\mathbb{Z}/4)$ is the multiplication by 2.

Remark 7.3. In fact, Theorem 7.2 also works if X is a paracompact space which has homotopy type of a locally finite 4-dimensional simplicial complex; for example, if X is a locally finite 4-dimensional CW complex. Indeed, this follows from the fact that $\mathrm{Mic}_m^+(X)$ is invariant under homotopy equivalences of paracompact spaces by the microbundle homotopy theorem (Theorem 3.1 in [Mil64]) for any $m > 0$.

For now, we focus on the case of 4-dimensional microbundles and explain the other cases in the end. Consider the map

$$\mathrm{BSTop}(4) \xrightarrow{(w_2, e, \tilde{p_1}, \mathrm{ks})} K(\mathbb{Z}/2,2) \times K(\mathbb{Z},4) \times K(\mathbb{Z},4) \times K(\mathbb{Z}/2,4) =: K$$

and denote it with α. Now consider the long exact sequence of homotopy groups associated to the fiber sequence

$$\mathrm{fib}(\alpha) \to \mathrm{BSTop}(4) \xrightarrow{\alpha} K.$$

The map $\pi_2 \,\mathrm{BSTop}(4) \to \pi_2 K$ is an isomorphism because $\mathrm{BSTop}(4) \xrightarrow{w_2} K(\mathbb{Z}/2,2)$ is the first stage of the Postnikov tower of $\mathrm{BSTop}(4)$. Therefore, the first possible nontrivial homotopy group of $\mathrm{fib}(\alpha)$ is $\pi_3 \,\mathrm{fib}(\alpha)$, which we prove to be isomorphic to $\mathbb{Z}/4$. From the long exact sequence of homotopy groups we have

$$\pi_4(\mathrm{BSTop}(4)) \xrightarrow{(e, \tilde{p_1}, \mathrm{ks})} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2 \to \pi_3 \,\mathrm{fib}(\alpha) \to 0.$$

To examine this cokernel we prove

Lemma 7.4. *The map*

$$\pi_4(\mathrm{BSTop}(4)) \to \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2$$
$$\xi \mapsto \left(e(\xi), \frac{\tilde{p_1}(\xi) + 2e(\xi)}{4}, \mathrm{ks}(\xi)\right)$$

is well-defined and is an isomorphism.

Proof. Firstly, it directly follows from Milnor's computations in [Mil56] that the map

$$\pi_4 \,\mathrm{BSO}(4) \xrightarrow{(e, \frac{p_1+2e}{4})} \mathbb{Z} \oplus \mathbb{Z}$$

is an isomorphism. Combining this isomorphism with Theorem 6.1 we have a commutative diagram of isomorphisms

$$\begin{array}{ccc} \pi_4 \,\mathrm{BSTop}(4) & \xrightarrow{\simeq} & \pi_4(\mathrm{BSO}(4)_{\leq 5} \times K(\mathbb{Z}/2,4)) \\ & \searrow^{(e, \frac{\tilde{p_1}+2e}{4}, \mathrm{ks})} & \downarrow{\scriptstyle (e', \frac{p_1'+2e'}{4}, \mathrm{ks})} \\ & & \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2, \end{array}$$

where $e', p_1' \in H^4(\mathrm{BSO}(4)_{\leq 4}, \mathbb{Z})$ are the classes corresponding to $e, p_1 \in H^4(\mathrm{BSO}(4), \mathbb{Z})$. □

By Lemma 7.4 we have a commutative diagram

$$\begin{array}{ccc}
\pi_4 \operatorname{BSTop}(4) & \xrightarrow{(e,\tilde{p_1},\mathrm{ks})} & \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}/2 \\
{\scriptstyle (e,\frac{\tilde{p_1}+2e}{4},\mathrm{ks})}\big\downarrow & \nearrow_{f} & \\
\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}/2 & &
\end{array}$$

with f given by the matrix

$$f = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and therefore $\pi_3 \operatorname{fib}(\alpha) \simeq \mathbb{Z}/4$. Now consider a Moore-Postnikov tower of $\operatorname{BSTop}(4) \xrightarrow{\alpha} K$

$$\begin{array}{ccccccc}
 & & & & \operatorname{MP}(\alpha)_4 & & \\
 & & & \nearrow & \big\downarrow & & \\
\operatorname{fib}(\alpha) & \longrightarrow & \operatorname{BSTop}(4) & \xrightarrow{\quad\alpha\quad} & K & \xrightarrow{\quad k\quad} & K(\mathbb{Z}/4,4).
\end{array}$$

Note also that $\pi_4 \operatorname{fib}(\alpha) = 0$, hence $\operatorname{MP}(\alpha)_4 = \operatorname{MP}(\alpha)_3$. By Lemma 4.16, the k-invariant $k \in H^4(K, \pi_3 \operatorname{fib}(\alpha))$ is equal to the transgression of the fundamental class $\gamma_3 \in H^3(\operatorname{fib}(\alpha), \pi_3 \operatorname{fib}(\alpha))$ in the Serre spectral sequence

$$H^p(K, H^q(\operatorname{fib}(\alpha), \pi_3 \operatorname{fib}(\alpha))) \implies H^{p+q}(\operatorname{BSTop}(4), \pi_3 \operatorname{fib}(\alpha)).$$

As usual, we consider the associated Serre exact sequence

$$\begin{array}{ccccccc}
H^3(\operatorname{BSTop}(4),\mathbb{Z}/4) & \longrightarrow & H^3(\operatorname{fib}(\alpha),\mathbb{Z}/4) & \longrightarrow & H^4(K,\mathbb{Z}/4) & \longrightarrow & H^4(\operatorname{BSTop}(4),\mathbb{Z}/4) \\
\simeq\big\downarrow & & \simeq\big\downarrow & & \simeq\big\downarrow & & \simeq\big\downarrow \\
\mathbb{Z}/2 & \longrightarrow & \mathbb{Z}/4 & \longrightarrow & \mathbb{Z}/4\langle \rho_4\iota_4^1, \rho_4\iota_4^2, P\gamma_2\rangle \oplus \mathbb{Z}/2\langle \gamma_4\rangle & \longrightarrow & \mathbb{Z}/4\langle \rho_4 e, \rho_4\tilde{p_1}\rangle \oplus \mathbb{Z}/2\langle \mathrm{ks}\rangle.
\end{array}$$

We claim that the kernel of the rightmost horizontal map is generated by the class $\rho_4\iota_4^2 - P\gamma_2 - \rho_4\iota_4^1$ (where ι_4^1, ι_4^2 denote the corresponding fundamental classes in $H^4(K(\mathbb{Z}\oplus\mathbb{Z},4),\mathbb{Z})$). It is enough to prove that there is a relation $\rho_4\tilde{p}_1 = Pw_2 + 2\rho_4 e$ in $H^4(\operatorname{BSTop}(4),\mathbb{Z}/4)$ to verify the claim. By Lemma 1 in [Mil58] this relation holds in cohomology of $\operatorname{BSO}(4)$. Moreover, we know that there is a relation

$$Pw_2 = a\rho_4\tilde{p}_1 + b\rho_4 e + c\,\mathrm{ks}$$

for some $a, b \in \mathbb{Z}/4, c \in \mathbb{Z}/2$. By mapping to $H^4(\operatorname{BSO}(4),\mathbb{Z}/4)$, we deduce that $a = 1, b = -2$. In addition, $c = 0$ because there are microbundles with the same $Pw_2, \tilde{p}_1, e$ but different ks over S^4 (Lemma 7.4).

Thus, by the exactness of the Serre sequence we conclude that $k = \pm(\rho_4\iota_4^2 - P\gamma_2 - 2\rho_4\iota_4^1)$.

Now we can prove Theorem 7.2.

proof of 7.2 (the case of 4-dimensional microbundles). Recall that there is a bijection $\operatorname{Mic}_4^+(X) \simeq [X, \operatorname{BSTop}(4)]$ by 2.11. Furthermore, the Moore-Postnikov tower induces a bijection $[X, \operatorname{BSTop}(4)] \to$

$[X, \mathrm{MP}(\alpha)_4]$ since $\mathrm{BSTop}(4) \to \mathrm{MP}(\alpha)_4$ is 5-connected and X has homotopy type of a 4-dimensional CW complex. The fiber sequence

$$\mathrm{MP}(\alpha)_4 \to K \xrightarrow{k} K(\mathbb{Z}/4, 4)$$

induces an exact sequence of pointed sets

$$\begin{array}{ccccccccc}
[X, \Omega K] & \xrightarrow{\Omega k_*} & [X, K(\mathbb{Z}/4, 3)] & \to & [X, \mathrm{MP}(\alpha)_4] & \longrightarrow & [X, K] & \xrightarrow{k_*} & [X, K(\mathbb{Z}/4, 4)] \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\
H^1(X, \mathbb{Z}/2) \times H^3(X, G) & \xrightarrow{\Omega k_*} & H^3(X, \mathbb{Z}/4) & \longrightarrow & \mathrm{Mic}_4^+(X) & \longrightarrow & H^2(X, \mathbb{Z}/2) \times H^4(X, G) & \xrightarrow{k_*} & H^4(X, \mathbb{Z}/4),
\end{array}$$

where $G \coloneqq \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2$ and $\Omega k : K \to K(\mathbb{Z}/4, 3)$ is the map induced by the functoriality of the loop space. Thus, the map

$$\mathrm{Mic}_4^+(X) \xrightarrow{(w_2, e, \tilde{p}_1, \mathrm{ks})} H^2(X, \mathbb{Z}/2) \times H^4(X, \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2)$$

is injective if an only if Ωk_* is surjective. Moreover, its image is equal to

$$\ker(k_*) = \{(a, b, c, d) : \rho_4 c = Pa + 2\rho_4 b\};$$

hence, it only remains to prove that Ωk_* is surjective whenever $H^4(X, \mathbb{Z})$ contains no 2-torsion. Write σ for the composition

$$\sigma : H^k(K(\mathbb{Z}/2, 2) \times K(G, 4), \mathbb{Z}/4) \to H^k(\Sigma(K(\mathbb{Z}/2, 1) \times K(G, 3)), \mathbb{Z}/4) \to$$

$$\xrightarrow{\Sigma} H^{k-1}(K(G, 3), \mathbb{Z}/4),$$

then $\sigma k = \Omega k$ (where Ωk is considered as an element of $H^3(K(G, 3), \mathbb{Z}/4)$) since the following diagram

$$\begin{array}{ccccc}
[K(A, 2) \times K(B, 4), K(C, 4)] & \xrightarrow{\Omega} & [\Omega(K(A, 2) \times K(B, 4)), \Omega K(C, 4)] & \xrightarrow{\simeq} & [K(A, 1) \times K(B, 3), \Omega K(C, 4)] \\
\downarrow & & \downarrow \simeq & & \uparrow \simeq \\
[\Sigma(K(A, 1) \times K(B, 3)), K(C, 4)] & \xrightarrow[\simeq]{} & [K(A, 1) \times K(B, 3), \Omega K(C, 4)] & \xleftarrow{\simeq} & [K(A, 1) \times K(B, 3), K(C, 3)]
\end{array}$$

commutes for any abelian groups A, B, C by naturality of the (Σ, Ω) adjunction. Therefore, we have

$$k = \pm(\rho_4 \iota_4^2 - P\gamma_2 - 2\rho_4 \iota_4^1) \in H^4(K(\mathbb{Z}/2, 2) \times K(\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2, 4), \mathbb{Z}/4),$$

$$\sigma k = \pm(\sigma\rho_4 \iota_4^2 - \sigma P\gamma_2 - 2\sigma\rho_4 \iota_4^1) = \pm(\rho_4 \iota_3^2 - \sigma P\gamma_2 - 2\rho_4 \iota_3^1) \in H^3(K(\mathbb{Z}/2, 1) \times K(\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2, 3), \mathbb{Z}/4).$$

Hence, $\Omega k_* = \sigma k_*$ is given by

$$H^1(X, \mathbb{Z}/2) \times H^3(X, \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2) \to H^3(X, \mathbb{Z}/4)$$

$$(a, b, c, d) \mapsto \pm(\rho_4 c - (\sigma P)_*(a) - 2\rho_4(b))$$

and it is surjective because the reduction mod 4 $\rho_4 : H^3(X, \mathbb{Z}) \to H^3(X, \mathbb{Z}/4)$ is surjective when there is no 2-torsion in $H^4(X, \mathbb{Z})$ by the Bockstein exact sequence. □

The cases of m-dimensional microbundles for $m = 3, m > 4$ follow by a similar computation of Moore-Postnikov towers, Smale's conjecture $\mathrm{BO}(3) \simeq \mathrm{BTop}(3)$ ([Hat83]) and Milgram's stable result on the Postnikov tower of BSTop – Theorem 5.9.

A Appendix

In this appendix we provide tables of cohomology groups of the relevant Eilenberg-Maclane spaces and elaborate on some of the cohomology operations which generate them.
We write

$$\iota_n \in H^n(K(\mathbb{Z}, n), \mathbb{Z}),$$

$$\gamma_n \in H^n(K(\mathbb{Z}/k, n), \mathbb{Z}/k),$$

for the fundamental classes;

$$\rho_k : H^n(-, \mathbb{Z}) \to H^n(-, \mathbb{Z}/k),$$

for the reduction mod k;

$$\beta_k : H^n(-, \mathbb{Z}/k) \to H^n(-, \mathbb{Z}),$$

for the Bockstein homomorphism;

$$\mathrm{Sq}^i : H^n(-, \mathbb{Z}/2) \to H^{n+i}(-, \mathbb{Z}/2),$$

for the Steenrod operations;

$$P : H^{2n}(-\mathbb{Z}/2) \to H^{4n}(-, \mathbb{Z}/4),$$

for the Pontryagin square (more information about this operation can be found in [MT08] Chapter 2).

Definition A.1. Let $I = (i_n, i_{n-1}, ..., i_0)$ be a sequence of natural numbers. I is called **admissible** if $i_k \geq 2i_{k-1}$ for all $k > 0$ and $i_0 > 0$. We also say that the empty sequence is admissible. **Excess** of I is the number $e(I) \coloneqq i_0 + (i_1 - 2i_0) + ... + (i_n - 2i_{n-1})$.

Denote $\mathrm{Sq}^I x \coloneqq \mathrm{Sq}^{i_n} \mathrm{Sq}^{i_{n-1}} ... \mathrm{Sq}^{i_0} x$ and $\mathrm{Sq}^\emptyset x \coloneqq x$. Serre proved the following theorem

Theorem A.2 ([MT08] Chapter 9)**.**

1. *$H^*(K(\mathbb{Z}, n), \mathbb{Z}/2)$ is a polynomial ring over $\mathbb{Z}/2$ with generators $\{\mathrm{Sq}^I \rho_2 \iota_n\}$ where I runs through all admissible sequences of excess less than n and $i_0 \neq 1$.*
2. *$H^*(K(\mathbb{Z}/2, n), \mathbb{Z}/2)$ is a polynomial ring over $\mathbb{Z}/2$ with generators $\{\mathrm{Sq}^I \gamma_n\}$ where I runs through all admissible sequences of excess less than n.*
3. *$H^*(K(\mathbb{Z}/2^m, n), \mathbb{Z}/2)$ is a polynomial ring over $\mathbb{Z}/2$ with generators $\{\mathrm{Sq}^{I_m} \gamma_n\}$ where I_m runs through all admissible sequences of excess less than n, $\mathrm{Sq}^{I_m} \coloneqq \mathrm{Sq}^I$ if $i_0 > 1$ and $\mathrm{Sq}^{I_m} \coloneqq \mathrm{Sq}^{i_n} \mathrm{Sq}^{i_{n-1}} ... \mathrm{Sq}^{i_1} \zeta_n$ with ζ_n being the generator of $H^{n+1}(K(\mathbb{Z}/2^m, n), \mathbb{Z}/2)$ otherwise.*

This provides us information about cohomology of these spaces with $\mathbb{Z}/2$ coefficients. Using the Serre spectral sequences of the fiber sequences

$$K(\mathbb{Z}, n) \to K(\mathbb{Z}, n) \to K(\mathbb{Z}/2, n),$$

$$K(A, n) \to * \to K(A, n+1),$$

one can also deduce some information about the integral cohomology of these Eilenberg-Maclane spaces. We provide a table with some of these groups

$H^i(K(\mathbb{Z}, n), \mathbb{Z}/2)$	$i = 3$	4	5	6	7	8
$n = 2$	0	$\langle \rho_2 \iota_2^2 \rangle$	0	$\langle \rho_2 \iota_2^3 \rangle$	0	$\langle \rho_2 \iota_2^4 \rangle$
3	$\langle \rho_2 \iota_3 \rangle$	0	$\langle \mathrm{Sq}^2 \rho_2 \iota_3 \rangle$	$\langle \rho_2 \iota_3^2 \rangle$	0	$\langle \rho_2 \iota_3 \mathrm{Sq}^2 \rho_2 \iota_3 \rangle$
4	0	$\langle \rho_2 \iota_4 \rangle$	0	$\langle \mathrm{Sq}^2 \rho_2 \iota_4 \rangle$	$\langle \mathrm{Sq}^3 \rho_2 \iota_4 \rangle$	$\langle \rho_2 \iota_4^2 \rangle$
5	0	0	$\langle \rho_2 \iota_5 \rangle$	0	$\langle \mathrm{Sq}^2 \rho_2 \iota_5 \rangle$	$\langle \mathrm{Sq}^3 \rho_2 \iota_5 \rangle$

$H^i(K(\mathbb{Z}/2, n), \mathbb{Z})$	$i = 3$	4	5	6	7	8
$n = 1$	0	$\mathbb{Z}/2\langle (\beta_2 \gamma_1)^2 \rangle$	0	$\mathbb{Z}/2\langle (\beta_2 \gamma_1)^3 \rangle$	0	$\mathbb{Z}/2\langle (\beta_2 \gamma_1)^4 \rangle$
2	$\mathbb{Z}/2\langle \beta_2 \gamma_2 \rangle$	0	$\mathbb{Z}/4\langle \beta_4 P \gamma_2 \rangle$	$\mathbb{Z}/2\langle (\beta_2 \gamma_2)^2 \rangle$	$\mathbb{Z}/2\langle \beta_2 (\gamma_2^3) \rangle$	$\mathbb{Z}/2\langle \beta_2 \gamma_2 \beta_4 P \gamma_2 \rangle$

$H^i(K(\mathbb{Z}/2, n), \mathbb{Z}/2)$	$i = 3$	4	5	6	7	8
$n = 1$	$\langle \gamma_1^3 \rangle$	$\langle \gamma_1^4 \rangle$	$\langle \gamma_1^5 \rangle$	$\langle \gamma_1^6 \rangle$	$\langle \gamma_1^7 \rangle$	$\langle \gamma_1^8 \rangle$
2	$\langle \mathrm{Sq}^1 \gamma_2 \rangle$	$\langle \gamma_2^2 \rangle$	$\langle \gamma_2 \mathrm{Sq}^1 \gamma_2, \mathrm{Sq}^2 \mathrm{Sq}^1 \gamma_2 \rangle$	$\langle \gamma_2^3, (\mathrm{Sq}^1 \gamma_2)^2 \rangle$	$\langle \gamma_2 \mathrm{Sq}^2 \mathrm{Sq}^1 \gamma_2,$ $\gamma_2^2 \mathrm{Sq}^1 \gamma_2 \rangle$	$\langle \gamma_2^4, \gamma_2 (\mathrm{Sq}^1 \gamma_2)^2,$ $\mathrm{Sq}^1 \gamma_2 \mathrm{Sq}^2 \mathrm{Sq}^1 \gamma_2 \rangle$

$H^i(K(\mathbb{Z}/12, n), \mathbb{Z}/2)$	$i = 3$	4	5	6	7	8
$n = 5$	0	0	$\langle \gamma_5 \rangle$	$\langle \zeta_5 \rangle$	$\langle \mathrm{Sq}^2 \gamma_5 \rangle$	$\langle \mathrm{Sq}^3 \gamma_5, \mathrm{Sq}^2 \zeta_5 \rangle$

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